A Perturbational Approach for Approximating Heterogeneous Agent Models

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What we do

- Canonical framework to study aggregate fluctuations
  - aggregate shocks + incomplete markets + het. agents (HA)
  - Challenge: equilibria are difficult to compute
- Existing methods often rely on 1st order appr. and MIT shocks
  - cannot study welfare from stabilization policies, risk, asset prices, portfolio choice
- This paper: proposes a novel method to approx HA economies
  - fast, efficient, and easy to implement
  - scalable to higher-order approximations

Key Insights

- Reformulate perturbations (at any order) using *directional derivatives*
  - values of directional derivatives remain low dimensional
  - solve small-dimensional linear systems with analytically solved closed-form coefficients
  - avoids refinements such as quadratic matrix equations and pruning
- Show the reformulation extends to HA economies
  - infinite-dimensional state
  - kinks in policy function
- Implementation needs only
  - steady-state objects using off-the-shelf algorithms
  - set of equations characterizing equilibrium

Plan

- Describe the method
  - First-order
  - Second-order
- Discuss differences from literature
  - State-space methods (Reiter, etc)
  - Sequence-space methods (Auclert et al, Boppart et al,etc)
- Applications
  - Speed/Accuracy
  - Welfare from stabilization dynamics
  - Effects of uncertainty
  - Portfolio problems

Notation

- $x_{i,t}$, X_t: individual and aggregate endogenous variables
  - $a_{i,t} \in x_{i,t}$, $A_t \in X_t$: pre-determined variables
  - equivalently for some selection matrices p, P as

$$a_{i,t} = \mathrm{p} x_{i,t}, \qquad A_t = \mathrm{P} X_t$$

- $\theta_{i,t}$, Θ_t: individual and aggregate exogenous variables
  - for this talk: $\theta_{i,t}, \Theta_t$ are scalars, follow AR(1) processes

$$\theta_{i,t} = \rho_\theta \theta_{i,t-1} + \varepsilon_{i,t}$$
$$\Theta_t = \rho_\Theta \Theta_{t-1} + \mathcal{E}_t$$

- $Y_t = \left[\Theta_t, \mathrm{P} X_{t-1}, X_t, \mathbb{E}_t X_{t+1}\right]^{\mathrm{T}}$: aggregate variables relevant for equilibrium in period t

Example: Krusell-Smith

- Households

$$\max_{\{c_{i,t},k_{i,t}\}_{t\geq 0}} \mathbb{E}_0 \sum_{t=0}^{\infty} \beta^t U(c_{i,t})$$

$$c_{i,t} + k_{i,t} = (1 + R_t - \delta)\, k_{i,t-1} + W_t \exp(\theta_{i,t})$$

$$k_{i,t} \geq 0$$

- Firms

$$\max_{N_t,K_t} \exp(\Theta_t)\, K_t^{\alpha} N_t^{1-\alpha} - W_t N_t - R_t K_t$$

- Market clearings

$$K_t = \int k_{i,t-1} di, \qquad N_t = \int \exp(\theta_{i,t})\, di$$

Mapping of KS economy

- Variables:

$$x_{i,t} = \left[k_{i,t}, c_{i,t}, \lambda_{i,t}, \zeta_{i,t}\right]^{\mathrm{T}}, \ a_{i,t} = k_{i,t}$$

$$X_t = \left[A_t, K_t, W_t, R_t\right]^{\mathrm{T}}, \ Y_t = \left[\Theta_t, A_{t-1}, X_t\right]^{\mathrm{T}}, \ A_t = \int a_{i,t} di$$

- Optimality conditions for agents with idiosyncratic risk:

$$\begin{aligned} c_{i,t} + k_{i,t} - (1 + R_t - \delta) k_{i,t-1} - W_t \exp(\theta_{i,t}) &= 0 \\ \lambda_{i,t} - (1 + R_t - \delta) u_c(c_{i,t}) &= 0 \\ u_c(c_{i,t}) + \zeta_{i,t} - \beta \mathbb{E}_t \lambda_{i,t+1} &= 0 \\ k_{i,t} \zeta_{i,t} &= 0 \end{aligned}$$

- All other conditions:

$$\begin{aligned} A_t - \int k_{i,t} di &= 0 \\ A_{t-1} - K_t &= 0 \\ R_t - \alpha \exp(\Theta_t) K_t^{\alpha-1} &= 0 \\ W_t - (1-\alpha) \exp(\Theta_t) K_t^{\alpha} &= 0 \end{aligned}$$

Canonical HA representation

- Optimality conditions of agents with idiosyncratic shocks:

$$F\left(\theta_{i,t}, a_{i,t-1}, x_{i,t}, \mathbb{E}_{i,t} x_{i,t+1}, Y_t\right) = 0 \text{ for all } i, t$$

  - inequality constraints that matter are folded as complementary slackness conditions

- All other equilibrium conditions:

$$G\left(\int x_{i,t} di, Y_t\right) = 0 \text{ for all } t$$

- Equilibrium: A stochastic sequence $\{X_t(\mathcal{E}^t), x_t(\mathcal{E}^t, \varepsilon^t)\}_{t,\mathcal{E}^t,\varepsilon^t}$ that satisfies equations F and G given initial conditions $\left(\{a_{i,-1}, \theta_{i,0}\}_i, A_{-1}\right)$

Recursive Representation

Let $Z=[A,\Theta,\Omega]^T$: aggregate state and (a,θ,Z) the individual states

- $\overline{x}(a,\theta,Z)$, $\overline{X}(Z)$, $\overline{\Omega}(Z)$ are indiv and agg policy functions
- $\overline{a}(a,\theta,Z)=\mathrm{p}\overline{x}(z,\theta,Z)$
- Recursive representation

$$F\left(a,\theta,\overline{x}(a,\theta,Z),\mathbb{E}_{\varepsilon,\mathcal{E}}\overline{x},\overline{Y}(Z)\right)=0$$

$$G\left(\int\overline{x}(\cdot,\cdot,Z)\,d\Omega,\overline{Y}(Z)\right)=0$$

$$\overline{\Omega}(Z)\langle a',\theta'\rangle=\int\int\iota\left(\overline{a}(a,\theta,Z)\leq a'\right)\iota(\rho_\theta\theta+\varepsilon\leq\theta')\mu(\varepsilon)\,d\varepsilon d\Omega\langle a,\theta\rangle\,.$$

where $\overline{Y}(Z)=\left[\Theta,A,\overline{X}(Z),\mathbb{E}_{\mathcal{E}}\overline{X}\right].^{\mathrm{T}}$

Perturbational approach

- Use the recursive (state-space) representation of equilibrium

- SS of economy without aggregate shocks, $Z^* = [0, A^*, \Omega^*]^{\mathrm{T}}$

- Perturb aggregate stochastic process by scalar $\sigma \geq 0$

$$\Theta_t = \rho_\Theta \Theta_{t-1} + \sigma \mathcal{E}_t$$

- Use Taylor expansions w.r.t. σ to find various orders of eqm approximations
  - technical conditions for stability and differentiability details
  - $\overline{X}$, $\overline{\Omega}$, $\overline{x}(a, \theta)$,...: policy functions at SS, $(Z, \sigma) = (Z^*, 0)$

Derivatives

- G_x, G_Y, G_{xY}, $\mathrm{F}_x(a,\theta)$, ...: derivatives of G and F, evaluated at SS
- $\overline{X}_Z$, $\overline{X}_{ZZ}$, $\overline{x}_Z(a,\theta)$, $\overline{Z}_Z$, ...: *(Frechet) derivatives* w.r.t. Z
- $\overline{x}_a(a,\theta), \overline{x}_{aZ}(a,\theta), ...$: derivatives w.r.t. a are *generalized functions*
  details

Directional derivatives: reminder

- $\overline{X}_Z$ is a large-dimensional object in HA economy
  - partial derivatives of $\overline{X}$ w.r.t. *each* dimension of Z
  - $\dim \overline{X}_Z = \dim X \times \dim Z$
- $\overline{X}_Z \cdot \hat{Z}$ is a value of this derivative in direction $\hat{Z}$:

$$\overline{X}_Z \cdot \hat{Z} = \lim_{\alpha \to 0} \frac{1}{\alpha} \left(\overline{X} \left(Z^* + \alpha \hat{Z}; 0 \right) - \overline{X} \left(Z^*; 0 \right) \right)$$

- $\overline{X}_Z \cdot \hat{Z}$ is always a small dimensional
  - $\dim \left(\overline{X}_Z \cdot \hat{Z} \right) = \dim X$
- Analogous observation applies to higher orders:
  - $\dim \overline{X}_{ZZ} = \dim X \times \dim Z \times \dim Z$
  - $\dim \left(\overline{X}_{ZZ} \cdot (\hat{Z}', \hat{Z}'') \right) = \dim X$

Our approach

- To approximate equilibrium, one needs to know not the whole derivative, but only its value in specific directions
  - e.g., 1^{st} order is approximated by

    $$\left\{\hat{X}_t\right\}_t := \left\{\overline{X}_Z \cdot \hat{Z}_t\right\}_t$$

    for a specific sequence of directions $\left\{\hat{Z}_t\right\}_t$
  - in HA economies, $\dim \overline{X}_{Z,t} << \dim \overline{X}_Z$
- Derive **analytically** expressions for directional values of those derivatives
- Show that this system is easy and quick to construct and solve numerically

1st order approx as directional derivatives

- To the 1^{st} order:

$$X_t\left(\mathcal{E}^t\right) = \overline{X} + \sum_{s=0}^{t} \hat{X}_{t-s}\mathcal{E}_s + O\left(\left\|\mathcal{E}\right\|^2\right),$$

where

$$\hat{Z}_0 := [1, 0, \mathbf{0}]^{\mathrm{T}}, \quad \hat{Z}_t := \overline{Z}_Z \cdot \hat{Z}_{t-1},$$
$$\hat{X}_t := \overline{X}_Z \cdot \hat{Z}_t.$$

- Economic intuition:
  - $\{\hat{Z}_t\}_t$ traces out LoM for Ω following a unit shock to Θ in period 0 (aka "MIT shock")
  - $\left\{\hat{X}_t\right\}_t$ is the impulse response to an MIT shock

Step 1: differentiate $G\left(\int \overline{x} d\Omega, \overline{Y}\right) = 0$

- First derivative of G in direction $\hat{Z}_t$:

$$\mathsf{G}_Y \hat{Y}_t + \mathsf{G}_x \left(\int \overline{x} d\Omega\right)_Z \cdot \hat{Z}_t = 0,$$

where

$$\hat{Y}_t = \left[\rho_{\Theta}^t, \mathsf{P}\hat{X}_{t-1}, \hat{X}_t, \hat{X}_{t+1}\right]^{\mathrm{T}}$$

and

$$\left(\int \overline{x} d\Omega\right)_Z \cdot \hat{Z}_t = \int \underbrace{\overline{x}_Z \cdot \hat{Z}_t}_{\hat{x}_t} d\Omega^* + \int \overline{x} d \underbrace{\Omega_Z \cdot \hat{Z}_t}_{\hat{\Omega}_t}$$

Step 1: differentiate $G\left(\int \overline{x} d\Omega, \overline{Y}\right) = 0$

- First derivative of G in direction $\hat{Z}_t$:

$$\mathrm{G}_Y \hat{Y}_t + \mathrm{G}_x \left(\int \overline{x} d\Omega\right)_Z \cdot \hat{Z}_t = 0,$$

where

$$\hat{Y}_t = \left[\rho_\Theta^t, \mathrm{P}\hat{X}_{t-1}, \hat{X}_t, \hat{X}_{t+1}\right]^{\mathrm{T}}$$

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$$\left(\int \overline{x} d\Omega\right)_Z \cdot \hat{Z}_t = \int \underbrace{\overline{x}_Z \cdot \hat{Z}_t}_{\hat{x}_t} d\Omega^* + \int \overline{x} d \underbrace{\Omega_Z \cdot \hat{Z}_t}_{\hat{\Omega}_t}$$

- We already know $d\Omega^*$, G_x, G_Y, P, ρ_Θ^t
- If we can express $\hat{x}_t$ and $\hat{\Omega}_t$ in terms of $\left\{\hat{X}_s\right\}_s$, we found a way to solve 1^{st} order

Step 2: differentiate $F\left(a,\theta,\overline{x},\mathbb{E}_{\varepsilon,\mathcal{E}}\overline{x},\overline{Y}\right)=0$

- First derivative of F in direction $\hat{Z}_t$:

$$\hat{x}_t\left(a,\theta\right)=\sum_{s=0}^{\infty}\mathsf{x}_s\left(a,\theta\right)\hat{Y}_{t+s}$$

where

$$\begin{aligned}\mathsf{x}_0\left(a,\theta\right)&=-\left(\mathsf{F}_x\left(a,\theta\right)+\mathsf{F}_{x^e}\left(a,\theta\right)\mathbb{E}\left[\overline{x}_a|a,\theta\right]\mathsf{p}\right)^{-1}\mathsf{F}_Y\left(a,\theta\right),\\ \mathsf{x}_{s+1}\left(a,\theta\right)&=-\left(\mathsf{F}_x\left(a,\theta\right)+\mathsf{F}_{x^e}\left(a,\theta\right)\mathbb{E}\left[\overline{x}_a|a,\theta\right]\mathsf{p}\right)^{-1}\mathsf{F}_{x^e}\left(a,\theta\right)\mathbb{E}\left[\mathsf{x}_s|a,\theta\right]\end{aligned}$$

- Intuition: x_s measures $\partial x_t/\partial Y_{t+s}$

Step 2: differentiate $F\left(a, \theta, \overline{x}, \mathbb{E}_{\varepsilon,\mathcal{E}}\overline{x}, \overline{Y}\right) = 0$

- First derivative of F in direction $\hat{Z}_t$:

$$\hat{x}_t\left(a, \theta\right) = \sum_{s=0}^{\infty} \mathrm{x}_s\left(a, \theta\right) \hat{Y}_{t+s}$$

  where

$$\begin{aligned} \mathrm{x}_0\left(a, \theta\right) &= -\left(\mathrm{F}_x\left(a, \theta\right) + \mathrm{F}_{x^e}\left(a, \theta\right) \mathbb{E}\left[\overline{x}_a | a, \theta\right] \mathrm{p}\right)^{-1} \mathrm{F}_Y\left(a, \theta\right), \\ \mathrm{x}_{s+1}\left(a, \theta\right) &= -\left(\mathrm{F}_x\left(a, \theta\right) + \mathrm{F}_{x^e}\left(a, \theta\right) \mathbb{E}\left[\overline{x}_a | a, \theta\right] \mathrm{p}\right)^{-1} \mathrm{F}_{x^e}\left(a, \theta\right) \mathbb{E}\left[\mathrm{x}_s | a, \theta\right] \end{aligned}$$

- Intuition: x_s measures $\partial x_t / \partial Y_{t+s}$
- RHS of (1) and (2) is known already $\Longrightarrow$ easy and fast to use these formulas to compute $\left\{\mathrm{x}_s\right\}_s$

Step 3: differentiate $\overline{\Omega}$

- Trio of operators, $\mathcal{M}$, $\mathcal{L}^{(a)}$, $\mathcal{I}^{(a)}$:

$$(\mathcal{M} \cdot y) \langle a', \theta' \rangle := \int \overline{\Lambda}(a', \theta', a, \theta) y \,(a, \theta)\, d\Omega^* \langle a, \theta \rangle \,,$$

$$\left(\mathcal{L}^{(a)} \cdot y\right) \langle a', \theta' \rangle := \int \overline{\Lambda}(a', \theta', a, \theta) \overline{a}_a \,(a, \theta)\, y \,(a, \theta)\, dad\theta,$$

$$\mathcal{I}^{(a)} \cdot y := \int \overline{x}_a(\theta, a) y \,(\theta, a)\, dad\theta.$$

Step 3: differentiate $\overline{\Omega}$

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$$\mathcal{I}^{(a)}\cdot y := \int \overline{x}_{a}(\theta,a)y\left(\theta,a\right)dad\theta.$$

- Intuition: suppose indiv. policy functions are perturbed by $\hat{a}_0\left(z,\theta\right)$
  - effect on agg. distribution in pd 1: $\frac{d}{d\theta}\hat{\Omega}_1 = -\mathcal{M}\cdot\hat{a}_0$
  - effect on agg. distribution in pd 2: $\frac{d}{d\theta}\hat{\Omega}_2 = \mathcal{L}^{(a)}\cdot\frac{d}{d\theta}\hat{\Omega}_1$

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$$\left(\mathcal{M}\cdot y\right)\left\langle a',\theta'\right\rangle := \int \overline{\Lambda}(a',\theta',a,\theta)y\left(a,\theta\right)d\Omega^{*}\left\langle a,\theta\right\rangle,$$

$$\left(\mathcal{L}^{(a)}\cdot y\right)\left\langle a',\theta'\right\rangle := \int \overline{\Lambda}(a',\theta',a,\theta)\overline{a}_{a}\left(a,\theta\right)y\left(a,\theta\right)dad\theta,$$

$$\mathcal{I}^{(a)}\cdot y := \int \overline{x}_{a}(\theta,a)y\left(\theta,a\right)dad\theta.$$

- Intuition: suppose indiv. policy functions are perturbed by $\hat{a}_0\left(z,\theta\right)$
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  - effect on agg. distribution in pd 2: $\frac{d}{d\theta}\hat{\Omega}_2 = \mathcal{L}^{(a)}\cdot\frac{d}{d\theta}\hat{\Omega}_1$

- Differentiate $\overline{\Omega}$ and plug definition of $\hat{Z}_t$:

$$\frac{d}{d\theta}\hat{\Omega}_{t+1} = \mathcal{L}^{(a)}\cdot\frac{d}{d\theta}\hat{\Omega}_t - \mathcal{M}\cdot\hat{a}_t$$

$$\int \overline{x}d\hat{\Omega}_t = -\mathcal{I}^{(a)}\cdot\frac{d}{d\theta}\hat{\Omega}_t$$

Put three steps together

- $\left\{\hat{X}_t\right\}_t$ solves

$$\mathrm{G}_Y \hat{Y}_t + \mathrm{G}_x \sum_{s=0}^{\infty} \mathrm{J}_{t,s} \hat{Y}_s = 0 \text{ for all } t$$

$$\hat{Y}_t = \left[\rho_\Theta^t, \mathrm{P}\hat{X}_{t-1}, \hat{X}_t, \hat{X}_{t+1}\right]^{\mathrm{T}}$$

where $\mathrm{J}_{0s} = \int \mathrm{x}_s d\Omega^*$

$$\mathrm{J}_{t,s} = \mathrm{J}_{t-1,s-1} + \mathcal{I}^{(a)} \cdot \left(\mathcal{L}^{(a)}\right)^{t-1} \cdot \mathcal{M} \cdot \mathrm{px}_s.$$

and $\mathrm{P}\hat{X}_{-1} = 0$, and $\lim_{t\to\infty} \hat{X}_t = 0$.

Higher-order approximations

- Second order requires tracking interactions/nonlinearity terms and precautionary motive terms
- Our approach from first order extends with minimal changes to higher orders
  - exactly the same steps to derive approx terms
  - almost the same mathematical form of equations
  - many 1st order terms get recycled for higher-order computations

Understanding the structure of second-derivatives

- Consider a composite function function $\overline{X} \circ \overline{Z}$ and perturb Z from $Z^* \rightarrow Z^* + \alpha' \hat{Z}' + \alpha'' \hat{Z}''$.

$$\lim_{\alpha'' \to 0} \lim_{\alpha' \to 0} \frac{1}{\alpha''} \frac{1}{\alpha'} \left[\overline{X} \left(\overline{Z} \left(Z^* + \alpha' \hat{Z}' + \alpha'' \hat{Z}'' \right) \right) - \overline{X} \left(\overline{Z} \left(Z^* \right) \right) \right]$$

$$= \underbrace{\overline{X}_Z}_{\text{1st order policy function}} \cdot \underbrace{\overline{Z}_{ZZ} \cdot (\hat{Z}', \hat{Z}'')}_{\text{2nd order state}} + \underbrace{\overline{X}_{ZZ}}_{\text{2nd order policy function}} \cdot \underbrace{(\overline{Z}_Z \cdot \hat{Z}', \overline{Z}_Z \cdot \hat{Z}'')}_{\text{1st order state}}.$$

- Similar logic for second-derivatives with respect to σ
  - exploit that first-order effects of σ are zero
  - σ appears directly in policy functions

- Relevant directions: In addition to directions $\left\{ \hat{Z}_t \right\}$ from before, we need to track $\left\{ \hat{Z}_{t,k} \right\}$ and $\left\{ \hat{Z}_{\sigma\sigma,t} \right\}$

$$\hat{Z}_{t,k} = \overline{Z}_Z \cdot \hat{Z}_{t-1,k-1} + \overline{Z}_{ZZ} \cdot \left(\hat{Z}_{t-1}, \hat{Z}_{k-1} \right)$$

$$\hat{Z}_{\sigma\sigma,t} = \overline{Z}_Z \cdot \hat{Z}_{\sigma\sigma,t-1} + \overline{Z}_{\sigma\sigma}$$

Second-order expansion

- Second-order approximation: X_t satisfies

$$X_t\left(\mathcal{E}^t\right) = \dots + \frac{1}{2}\left(\sum_{s=0}^{t}\sum_{m=0}^{t}\hat{X}_{t-s,t-m}\mathcal{E}_s\mathcal{E}_m + \hat{X}_{\sigma\sigma,t}\right) + O\left(\|\mathcal{E}\|^3\right),$$

- $\left\{\hat{X}_{t,k}\right\}_{t,k}$ and $\left\{\hat{X}_{\sigma\sigma,t}\right\}_t$ are defined by

$$\hat{X}_{t,k} := \overline{X}_Z \cdot \hat{Z}_{t,k} + \overline{X}_{ZZ} \cdot \left(\hat{Z}_t, \hat{Z}_k\right)$$

$$\overline{X}_{\sigma\sigma,t} := \overline{X}_Z \cdot \hat{Z}_{\sigma\sigma,t} + \overline{X}_{\sigma\sigma}$$

with $\hat{Z}_{0,k} = \hat{Z}_{t,0} = \hat{Z}_{\sigma\sigma,0} = 0$.

- Automatically "pruned" simulation paths (Lombardo, Uhlig)

Need to find $\{\hat{X}_{t,t+k}\}_t$ and $\{\hat{X}_{\sigma\sigma,t}\}_t$

- For second-order interaction terms differentiate mapping G
  - once in direction $\hat{Z}_{t,k}$ and twice in directions $\hat{Z}_t, \hat{Z}_k$
- For precautionary terms differentiate mapping G in
  - once in direction $\hat{Z}_{\sigma\sigma,t}$ and twice w.r.t σ
- End up with unknown objects like $\left\{\hat{x}_{t,k}, \hat{\Omega}_{t,k}\right\}, \left\{\hat{x}_{\sigma\sigma,t}, \hat{\Omega}_{\sigma\sigma,t}\right\}$
  - use mapping F and $\overline{\Omega}$ to express them in terms of $\{\hat{X}_{t,t+k}\}_t$, $\{\hat{X}_{\sigma\sigma,t}\}_t$ and other terms that are known

Intermediate terms: $\left\{\hat{x}_{t,k}\hat{\Omega}_{t,k}\right\},\left\{\hat{x}_{\sigma\sigma,t}\hat{\Omega}_{\sigma\sigma,t}\right\}$

- $\hat{x}_{t,t+k}\left(a,\theta\right)$ and $\hat{x}_{\sigma\sigma,t}\left(a,\theta\right)$ satisfy

$$\hat{x}_{t,t+k}\left(a,\theta\right)=\underbrace{\sum_{s=0}^{\infty}\mathrm{x}_{s}\left(a,\theta\right)\hat{Y}_{t+s,t+k+s}}_{\text{1st order policy,2 nd orde state}}+\underbrace{\mathrm{x}_{t,t+k}\left(a,\theta\right)}_{\text{2nd order policy, first-order states}},$$

$$\hat{x}_{\sigma\sigma,t}\left(a,\theta\right)=\sum_{s=0}^{\infty}\mathrm{x}_{s}\left(a,\theta\right)\hat{Y}_{\sigma\sigma,t+s}+\mathrm{x}_{\sigma\sigma}\left(a,\theta\right).$$

- $\frac{d}{d\theta}\hat{\Omega}_{t,t+k}$ and $\frac{d}{d\theta}\hat{\Omega}_{\sigma,\sigma,t+1}$ satisfy

$$\frac{d}{d\theta}\hat{\Omega}_{t+1,t+k+1}=\underbrace{\mathcal{L}^{(a)}\cdot\frac{d}{d\theta}\hat{\Omega}_{t,t+k}-\mathcal{M}\cdot\hat{a}_{t,t+k}}_{\text{1st order policy, 2nd orde state}}+\underbrace{\frac{d}{da}\mathrm{c}_{t,t+k}-\mathrm{b}_{t,t+k}}_{\text{2nd order policy, first-order states}},$$

$$\frac{d}{d\theta}\hat{\Omega}_{\sigma,\sigma,t+1}=\mathcal{L}^{(a)}\cdot\frac{d}{d\theta}\hat{\Omega}_{\sigma,\sigma,t}-\mathcal{M}\cdot\hat{a}_{\sigma,\sigma,t}.$$

with closed-form expressions for all coefficients

2nd order solution

- $\{\hat{X}_{t,t+k}\}_t$ satisfies

$$\mathsf{G}_Y \hat{Y}_{ZZ,t,t+k} + \hat{\mathsf{G}}_{t,t+k} + \mathsf{G}_x \sum_{s=0}^{\infty} \mathsf{J}_{t,s} \hat{Y}_{s,s+k} + \mathsf{G}_x \color{red}{\mathsf{H}_{t,t+k}} = 0 \text{ for all } t,$$

where $\hat{\mathsf{G}}_{t,t+k} = \mathsf{G}_{\mathcal{Y}\mathcal{Y}} \cdot (\hat{\mathcal{Y}}_t, \hat{\mathcal{Y}}_{t+k})$ and $\hat{\mathcal{Y}}_t = \left[\hat{Y}_t, \left(\int \overline{x} d\Omega\right)_Z \cdot \hat{Z}_t\right]^{\mathrm{T}}$.

- $\{\hat{X}_{\sigma\sigma,t}\}_t$ satisfies

$$\mathsf{G}_Y \hat{Y}_{\sigma\sigma,t} + \mathsf{G}_x \sum_{s=0}^{\infty} \mathsf{J}_{t,s} \hat{Y}_{\sigma\sigma,s} + \mathsf{G}_x \color{red}{\mathsf{H}_{\sigma\sigma,t}} = 0 \text{ for all } t.$$

- Boundary conditions

$$\lim_{t\to\infty} \hat{X}_{t,t+k} = 0, \quad \lim_{t\to\infty} \hat{X}_{\sigma\sigma,t} - \hat{X}_{\sigma\sigma,t-1} = 0.$$

- Terms in red: new, calculated similarly to terms like G_x, $\mathsf{J}_{t,s}$ and derived in closed-form

Numerical Implementation

- User inputs: SS policy functions (splines) and equations describing competitive equilibrium
- First-order: need G_x, G_Y and $\{\mathsf{J}_{t,s}\}_{t,s}$ to solve

$$\mathsf{G}_Y \hat{Y}_t + \mathsf{G}_x \sum_{s=0}^{\infty} \mathsf{J}_{t,s} \hat{Y}_s = 0 \text{ for all } t$$

  - G_x, G_Y automatically differentiate G and evaluate at SS
  - Since $\mathsf{J}_{t,s} = \mathsf{J}_{t-1,s-1} + \mathcal{I}^{(a)} \cdot \left(\mathcal{L}^{(a)}\right)^{t-1} \cdot \mathcal{M} \cdot \mathsf{px}_s$ need
    - Operators $\mathcal{I}^{(a)}$, $\mathcal{L}^{(a)}$ and $\mathcal{M}$: sparse matrices using SS transition matrix
    - Coeffs $\{\mathsf{x}_t\}_t$: recursively using linear algebra with pre-computed basis matrices
  - Solve the linear system $\{\hat{X}_t\}_{t=0}^{T}$ by truncation
- Second-order follows similarly
  - need to keep track of kinks as the first-derivative function jumps

details

Comparison to other perturbation approaches

- Comparison to Reiter (2009):
  - ours is much faster to the 1^{st} order (no need to compute $\overline{X}_Z$, $\overline{Z}_Z$)
  - ours generalizes to higher orders (histogram method fails beyond 1^{st} order) details
- Comparison to Auclert et al (2022):
  - 1^{st} order: we construct same theoretical objects in a different way $\rightarrow$ two methods produce the same solution as grid size goes to zero
  - ours a bit faster (closed form expressions give $\mathrm{x}_s = \partial x_t / \partial Y_{t+s}$)
  - ours generalizes to higher orders (histogram method fails beyond 1^{st} order, MIT shocks do not capture effects of risk)
- Comparison to Bhandari et al (2021):
  - they use similar perturbational techniques but with respect to $(\mathcal{E}, \varepsilon)$
  - their approach does not work when policy functions $\overline{x}$ have kinks
- Bilal (2023) and Alvarez et al (2023): Continuous-time settings
  - share with us the use of analytic derivatives and linear operators
  - underlying mathematics quite different and the relative advantages vary by application
  - dont allow for endogenous kinks, portfolio or heteroskedastic shocks

Numerical illustrations

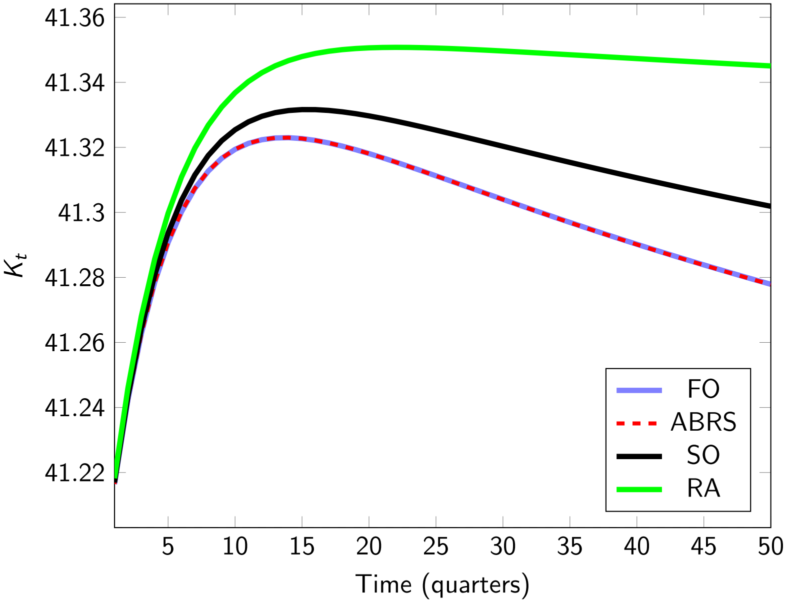
- Calibrated KS economy with capital adjustment costs
  - adjustment costs match volatility of equity returns
- Some experiments
  1. computational speed, comparison with alternatives
  2. optimal portfolios of stocks and bonds in KS economy
  3. effect of stochastic volatility shocks
  4. welfare of stabilization policies, failure of histogram method

1. Comparisons

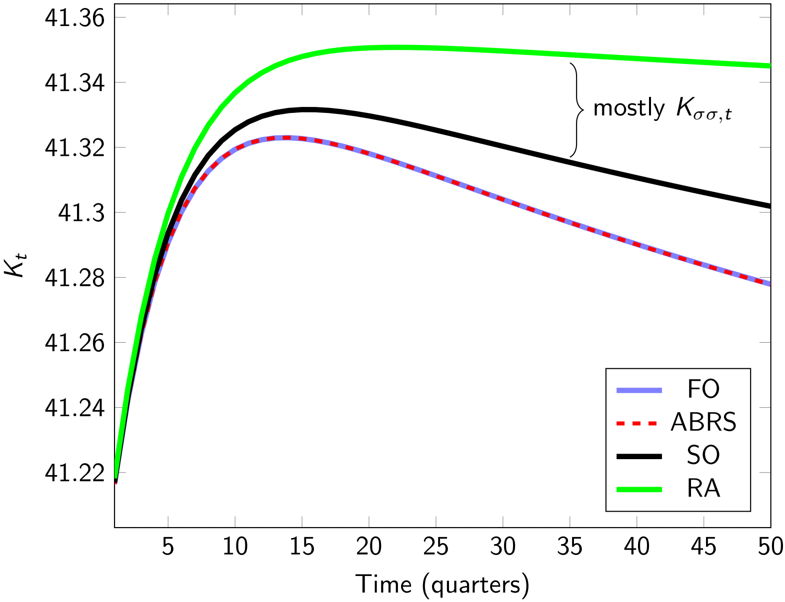
Table: COMPUTATIONAL SPEED: FIRST AND SECOND ORDER

First Order		Second Order	
Step	Time	Step	Time (ZZ)
		Additional First-Order Terms	0.63s
Compute $\{x_s\}$	0.11s	Compute $\{x_{t,k}\}$ and $\{x_{\sigma\sigma}\}$	1.28s
Compute $\mathcal{M}$, $\mathcal{L}$ and $\{\hat{a}_t\}_t$	0.02s	Compute $\{b_{t,k}, c_{t,k}\}$ and $\{b_{\sigma\sigma}\}$	0.27s
Compute $\{J_{t,s}\}_{t,s}$	0.21s	Compute $H_{t,k}$ and $H_{\sigma\sigma,t}$	0.18s
Compute $\{\hat{X}_t\}_t$	0.10s	Compute $\{\overline{\hat{X}}_{t,k}\}_{t,k}$, $\{\overline{\hat{X}}_{\sigma\sigma,t}\}_t$	0.20s
Total	0.44s		2.57s
ABRS	0.51s		

1. Simulations: $\mathcal{E}^t = (1, 0, 0, 0, \ldots)$



1. Simulations: $\mathcal{E}^t = (1, 0, 0, 0, \ldots)$



2. Household Portfolios in GE

- Consider a two asset extension – risky capital and risk-free bond – of Krusell and Smith economy
- Cannot be solved directly with conventional perturbational techniques:
  - optimal portfolios are undetermined at the zeroth order
  - optimal portfolios depend on second-order properties (risk) but already affect first-order impulse responses
- Before we solved for $\{\hat{X}_t\}$ so that $G()$ was satisfied by characterizing individual behavior $\hat{x}_t$ and aggregation $\hat{\Omega}_t$
  - Same approach but now $\hat{x}_t$ will depend on hh portfolios

2. Household Portfolios in GE

- Merton-Samuelson : portfolios (expected excess returns, hedging)

$$0 = \mathbb{E}_{\varepsilon,\mathcal{E}}\left[\lambda R^x\right] \approx \underbrace{\mathbb{E}_{\varepsilon}\left[\overline{\lambda}|a,\theta\right]\overline{R}^x_{\sigma\sigma}}_{\text{expected excess returns}} + \underbrace{\mathbb{E}_{\varepsilon}\left[\hat{\lambda}\left(\text{portfolio}, \left\{\hat{X}_t\right\}\right)\hat{R}^x var\left(\mathcal{E}\right)|a,\theta\right]}_{\text{hedging}}$$

- Conditional on distribution of hh portfolios
  - Expected excess returns, $\overline{R}^x_{\sigma\sigma}$ clear market for risky capital
  - $\hat{R}^x_t \in \hat{X}_t$ solve $G\left(\right)$ as before
  - Seemingly big fixed point!

- Main result: $\{\hat{X}_t\}_t$ are the solution to

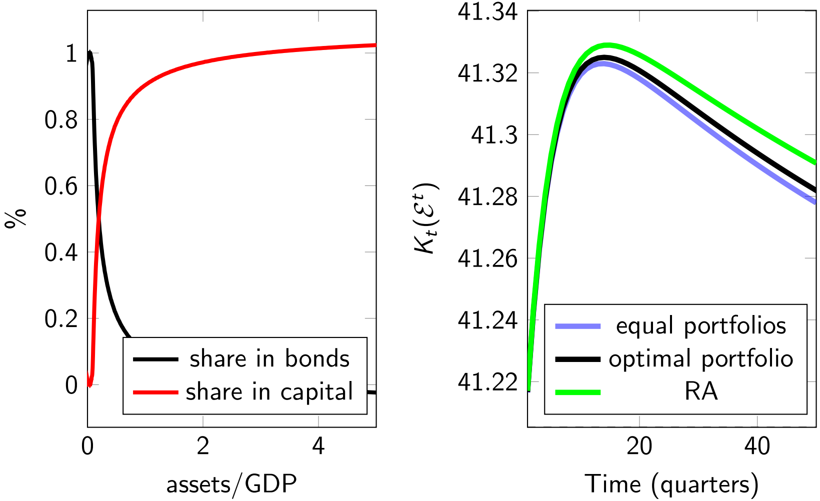
$$\mathrm{G}_Y\hat{Y}_t + \mathrm{G}_x\sum_{s=0}^{\infty}\left(\mathrm{J}_{t,s} + \mathrm{J}^{PP}_{t,s}\right)\hat{Y}_s = 0 \text{ for all } t$$

where $\mathrm{J}^{PP}_{t,s}$ has a similar structure to $\mathrm{J}_{t,s}$ can be derived in closed form.

  - by product: holding of risky assets $\overline{k}(a,\theta)$

2. Household Portfolios in GE

Figure: PORTFOLIOS



3. Effects of Uncertainty

- Large empirical literature about macroeconomic uncertainty
- What are the aggregate and distributional effects of uncertainty?
- Calibrate uncertainty shock to capture changes in VIX
- Modify TFP process to $\mathcal{E}_t = \sqrt{1 + \Upsilon_{t-1}} \mathcal{E}_{\Theta,t}$ where

$$\Upsilon_t = \rho_\Upsilon \Upsilon_{t-1} + \mathcal{E}_{\Upsilon,t}.$$

3. Effects of Uncertainty

- Traditional perturbation methods (see Villaverde) scale $\sigma\left[\mathcal{E}_{\Theta,t}, \mathcal{E}_{\Upsilon,t}\right]$
  - effects of $\mathcal{E}_{\Upsilon,t}$ only show up at **third-order**
- Our approach: Scale $\sigma\mathcal{E}_t$
  - "point" of approximation $Z^*(\Upsilon) = \left[0, \mathrm{P}\overline{X}, \Upsilon\right]$
  - effects of time-varying vol at **second-order**
  - directions $\hat{Z}_{\sigma\sigma,t}$ become stochastic directions $\hat{Z}_{\sigma\sigma,t}\left(\mathcal{E}^t_\Upsilon\right)$
- Main result:

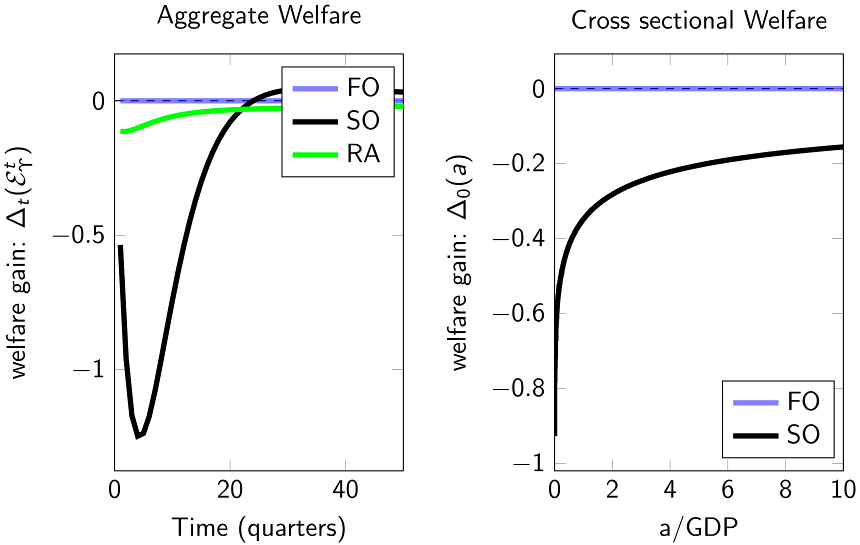
$$\hat{X}^{\Upsilon}_{\sigma\sigma,t}(\mathcal{E}^t_\Upsilon) = \overline{X}_Z \cdot \hat{Z}^{\Upsilon}_{\sigma\sigma,t}(\mathcal{E}^t_\Upsilon) = \hat{X}_{\sigma\sigma,t} + \sum_{s=0}^{t} \hat{X}^{SV}_{\sigma\sigma,t-s}\mathcal{E}_{\Upsilon,s} +$$

and sequence $\{\hat{X}^{SV}_{\sigma\sigma,t}\}_t$ satisfies

$$\mathrm{G}_Y \hat{Y}^{SV}_{\sigma\sigma,t} + \mathrm{G}_x \sum_{s=0}^{\infty} \mathrm{J}_{t,s} \hat{Y}^{SV}_{\sigma\sigma,s} + \mathrm{G}_x \mathrm{H}^{SV}_{\sigma\sigma,t} = 0 \text{ for all } t,$$

where $\mathrm{H}^{SV}_{\sigma\sigma,t}$ can be constructed recursively (see paper)

3. Welfare impact ↑of VIX by 5X



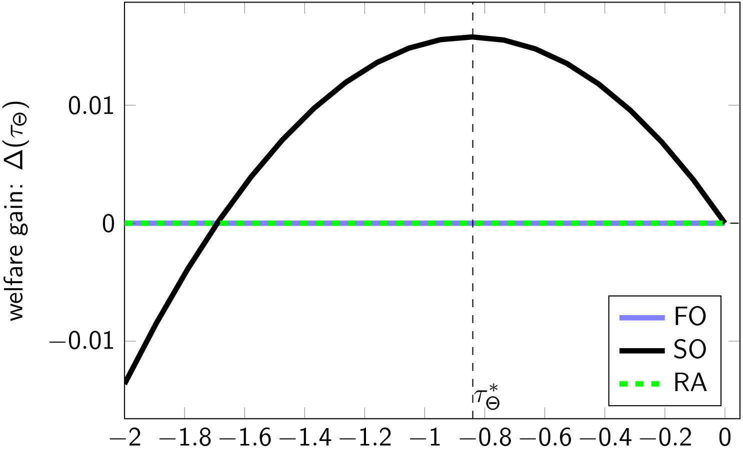
4. Stabilization Policy

- Simple model of stabilization policy: choose optimal τ_Θ in

$$\tau_t = \tau_\Theta \Theta_t$$

- Stabilization policy is a second order question
  - τ_Θ has no effect on welfare to the first order
- Compare answers if we tried to track distribution using the histogram method

4. Stabilization Policy



- Optimal τ^*_Θ requires raises taxes by 0.84pp for 1% drop in aggregate TFP
- Histogram method implies incorrect level and gradient of welfare $\tau^{*,hist}_\Theta = -1.04pp$

5. Transition Dynamics

- So far initial conditions = steady-state...Now consider

$$(A_{-1}, \Omega_0) \neq (A^*, \Omega^*)$$

- Useful to study tax-reforms and other permenant changes in primitives
- Represent the deterministic path using directional derivatives
  - almost the same calculations!

5. Transition Dynamics

Before... | For transition dynamics...

Approximation

$$X_t\left(\mathcal{E}^t\right) \approx \overline{X} + \sum_{s=0}^{t} \hat{X}_{t-s}\mathcal{E}_s \qquad \hat{X}_t := \overline{X}_Z \cdot \hat{Z}_t$$

$$\mathbb{E}_0 X_t \approx \overline{X} + \hat{X}_t^{TD} \qquad \hat{X}_t^{TD} := \overline{X}_Z \cdot \hat{Z}_t^{TD}$$

Relevant directions

$$\hat{Z}_t := \overline{Z}_Z \cdot \hat{Z}_{t-1} \qquad \hat{Z}_0 := [1, 0, \mathbf{0}]^{\mathrm{T}}$$

$$\hat{Z}_t^{TD} := \overline{Z}_Z \cdot \hat{Z}_{t-1}^{TD} \qquad \hat{Z}_0^{TD} := \left[0, A_{-1} - A^*, \Omega_{-1} - \Omega^*\right]^{\mathrm{T}}$$

Solution

$\left\{\hat{X}_t\right\}$ solved

$$\mathsf{G}_Y \hat{Y}_t + \mathsf{G}_x \sum_{s=0}^{\infty} \mathsf{J}_{t,s} \hat{Y}_s = 0$$

$\left\{\hat{X}_t^{TD}\right\}$ solves

$$\mathsf{G}_Y \hat{Y}_t^{TD} + \mathsf{G}_x \sum_{s=0}^{\infty} \mathsf{J}_{t,s} \hat{Y}_s^{TD} + \mathsf{G}_x \mathsf{J}_t^{TD} = 0$$

$$\mathsf{J}_t^{TD} = \mathcal{I}^{(a)} \cdot \left(\mathcal{L}^{(a)}\right)^{t-1} \cdot \left(-\frac{d}{d\theta}\hat{\Omega}_0^{TD}\right)$$

Conclusions

- Tool for higher order approximations of heterogeneous agent models
- Opens up plenty of interesting applications in macro and finance

Appendix

Non-negativity

- Occasionally borrowing limits impose inequality constraints on choices and multipliers

$$k_{i,t} \geq 0, \zeta_{i,t} \geq 0.$$

- Since these constraints "can" bind, apriori they may impose restrictions on the local behavior of policy functions $\left(k_Z\left(a,\theta,Z;\sigma\right)\cdot\hat{Z}, \zeta_Z\left(a,\theta,Z;\sigma\right)\cdot\hat{Z}\right)$ derived from other optimality conditions.
- However completementary slackness conditions restrict those local properties

$$\overline{k}\left(a,\theta\right) > 0 \implies \zeta_Z\left(a,\theta,Z;\sigma\right)\cdot\hat{Z} = 0$$

$$\overline{\zeta}\left(a,\theta\right) > 0 \implies k_Z\left(a,\theta,Z;\sigma\right)\cdot\hat{Z} = 0$$

- Thus when $\overline{k}\left(a,\theta\right) > 0$ (symmetric argument for the other case)
  - $k\left(a,\theta,Z;\sigma\right)$ locally satisfies $k_{i,t} \geq 0$
  - $\zeta\left(a,\theta,Z;\sigma\right)$ is locally unaffected so satisfies $\zeta_{i,t} \geq 0$

Technical Assumptions

Let $\overline{Z}_t := \underbrace{\overline{Z}(\overline{Z}(....\overline{Z}}_{t \text{ times}}(Z_0)))$.

1. $\mathcal{E}_t$ is mean-zero i.i.d with bounded support
2. $\overline{X}(Z;\sigma)$ is sufficiently differentiable at $(Z^*,0)$;
3. $\lim_{t\to\infty}\overline{Z}_t(Z_0) = Z^*$ for all Z_0 in a neighborhood of Z^*.
4. $\overline{x}(a,\theta,Z;\sigma)$ is continuous and piecewise sufficiently differentiable at $(Z^*,0)$ for all (a,θ) with finitely many points of non-differentiability
5. The marginal distribution $\int \Omega^* d\theta$ has a finite number of mass-points $\{a_n^*\}_n$

back

Generalized functions

- Linear functionals

$$d[\phi] = \int d(x)\,\phi(x)\,dx$$

example: Dirac delta δ is a genralized function with

$$\delta[\phi] = \int \phi(x)\,\delta(x)\,dx = \phi(0)$$

- Convenient to handle kinks

$$d[\phi] = \int \underbrace{d(x)}_{\text{may have kinks}} \times \underbrace{\phi(x)}_{\text{“nice” functions}}$$

- Easy rules for differentiation

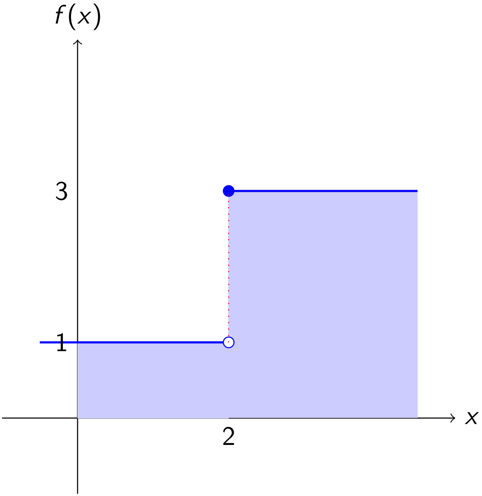
$$d'[\phi] = -d[\phi']$$

example: Indicator function that is, $d(x) = 1 \quad x \geq 0$ and $d(x) = 0$ otherwise.

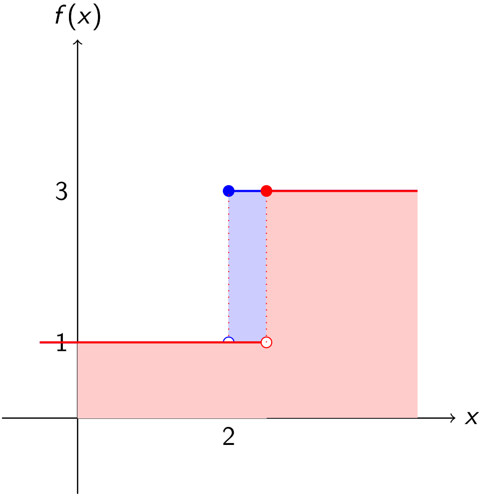
$$d' = -\int_0^\infty \phi'(x) = \phi(0) = \int \delta(x)\,\phi(x) = \delta$$

back

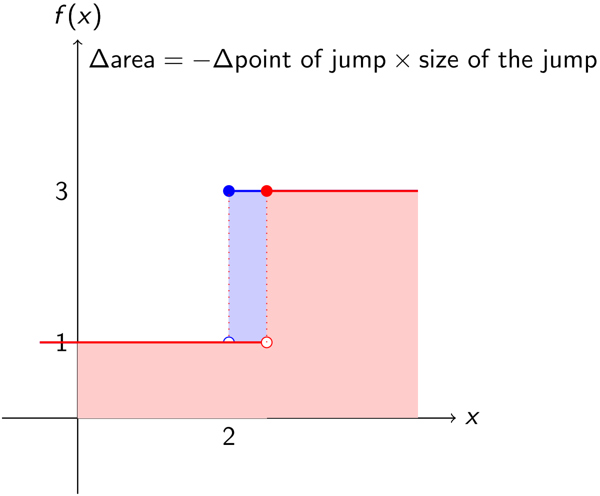
How Kinks Work



How Kinks Work



How Kinks Work



How Kinks Work

- Policy functions have kinks
  - Suppose the $\overline{x}(a, \theta, Z)$ has a kink at $a = \overline{\kappa}(\theta, Z)$
  - derivatives of policy functions will have jumps
- When the aggregate state changes

$$\overline{x}_{aZ}(a,\theta) \cdot \hat{Z} = \underbrace{\mathring{\overline{x}}_{aZ}(a,\theta) \cdot \hat{Z}}_{\text{classical}} - \underbrace{\delta(a - \overline{\kappa}(\theta))\overline{x}_a^{\Delta}(\theta)\overline{\kappa}_Z(\theta) \cdot \hat{Z}}_{\text{generalized function}}$$

- So integrals

$$\iint \overline{x}_{aZ}(a,\theta) \cdot \hat{Z}\omega(a,\theta)da d\theta$$
$$= \underbrace{\iint \mathring{\overline{x}}_{aZ}(a,\theta) \cdot \hat{Z}\omega(a,\theta)da d\theta}_{\text{classical}} - \int \underbrace{\omega(\overline{\kappa}(\theta),\theta)}_{\text{density at kink}} \times \underbrace{\overline{x}_a^{\Delta}(\theta)}_{\text{size of jump}} \times \underbrace{\overline{\kappa}_Z(\theta)}_{\text{movement in kink}} \cdot \hat{Z} d\theta$$

- Terms like these matter at second-order and need to be explicitly tracked back

Histogram method (Review)

- Histogram (bins,mass points) to approximate Ω
  - grid $\{\mathrm{z}_i\}_{i=0}^N$ represent midpoints of bins
  - $\{\omega_i^z\}$ mass at points $\{\mathrm{z}_i\}_{i=0}^N$

- Functions $\{\mathcal{P}^i(\cdot)\}$ so for $z \in [\mathrm{z}_i, \mathrm{z}_{i+1}]$ only non-zero values

$$\mathcal{P}^i(z) = \frac{\mathrm{z}_{i+1} - z}{\mathrm{z}_{i+1} - \mathrm{z}_i}, \quad \mathcal{P}^{i+1}(z) = \frac{z - \mathrm{z}_i}{\mathrm{z}_{i+1} - \mathrm{z}_i}.$$

  - $\mathcal{P}^i(z)$: the probability z is assigned to bin with midpoint z_i.

- Applications: Linear approximates for aggregates and LOM
  - $\int x(z,\theta)\, d\Omega \approx \int \sum_i x(\mathrm{z}_i,\theta)\, \omega_i^z\, dF(\theta)$
  - $\tilde{\omega}_j^z(\Theta,\omega) \approx \sum_i \omega_i^z \int \mathcal{P}^j(\tilde{z}(\mathrm{z}_i,\theta,\Theta,\omega^z))\, dF(\theta)$

- Standard approach: Differentiate after applying discretizing using histogram method

Why does Histogram method fail? Simple Example

Histogram method approximates $f(z) \approx \sum_{i=0}^{N} \mathcal{P}^i(z) f(\mathrm{z}_i)$. Now...

- Expand LHS $f(z+\hat{z})$

$$f(z) + f'(z)\hat{z} + \frac{1}{2} f''(z)\hat{z}^2 + o\left(\hat{z}^2\right)$$

- Expand RHS $\sum_{i=0}^{N} \mathcal{P}^i(z+\hat{z}) f(\mathrm{z}_i)$

$$\sum_{i=0}^{N} \mathcal{P}^i(z) f(\mathrm{z}_i) + \sum_{i=0}^{N} \mathcal{P}^i_z(z) f(\mathrm{z}_i)\hat{z} + \frac{1}{2}\sum_{i=0}^{N} \mathcal{P}^i_{zz}(z) f(\mathrm{z}_i)\hat{z}^2 + o\left(\hat{z}^2\right)$$

- Now take limits as $N \to \infty$
    - zeroth order $\sum_{i=0}^{N} \mathcal{P}^i(z) f(\mathrm{z}_i) \to f(z)$

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- Expand RHS $\sum_{i=0}^{N} \mathcal{P}^i(z+\hat{z}) f(\mathrm{z}_i)$

$$\sum_{i=0}^{N} \mathcal{P}^i(z) f(\mathrm{z}_i) + \sum_{i=0}^{N} \mathcal{P}_z^i(z) f(\mathrm{z}_i)\hat{z} + \frac{1}{2} \sum_{i=0}^{N} \mathcal{P}_{zz}^i(z) f(\mathrm{z}_i)\hat{z}^2 + o\left(\hat{z}^2\right)$$

- Now take limits as $N \to \infty$
  - first order $\sum_{i=0}^{N} \mathcal{P}_z^i(z) f(\mathrm{z}_i)\hat{z} = \frac{f\left(\mathrm{z}_{i+1}\right) - f(\mathrm{z}_i)}{\mathrm{z}_{i+1} - \mathrm{z}_i} \to f'(z)\hat{z}$

Why does Histogram method fail? Simple Example

Histogram method approximates $f(z) \approx \sum_{i=0}^{N} \mathcal{P}^i(z) f(\mathrm{z}_i)$. Now...

- Expand LHS $f(z+\hat{z})$

$$f(z) + f'(z)\hat{z} + \frac{1}{2} f''(z)\hat{z}^2 + o\left(\hat{z}^2\right)$$

- Expand RHS $\sum_{i=0}^{N} \mathcal{P}^i(z+\hat{z}) f(\mathrm{z}_i)$

$$\sum_{i=0}^{N} \mathcal{P}^i(z) f(\mathrm{z}_i) + \sum_{i=0}^{N} \mathcal{P}_z^i(z) f(\mathrm{z}_i)\hat{z} + \frac{1}{2}\sum_{i=0}^{N} \mathcal{P}_{zz}^i(z) f(\mathrm{z}_i)\hat{z}^2 + o\left(\hat{z}^2\right)$$

- Now take limits as $N \to \infty$
  - second order $\sum_{i=0}^{N} \mathcal{P}_{zz}^i(z) f(\mathrm{z}_i)\hat{z}^2 = 0 \nrightarrow f''(z)\hat{z}^2$

Why does Histogram method fail?

- Tractability of histogram methods come from "uniform" lotteries
  - preserves mass and conditional means

$$\sum_i \mathcal{P}^i(z) = 1$$

$$\sum_i \mathcal{P}^i(z)\,\mathrm{z}_i = z$$

  - which works for first-order but not higher in presence of curvature

- Our approach discretizes after differentiating
  - approximates $f''(z)\,\hat{z}$ instead of $\sum_{i=0}^{N} \mathcal{P}^i_{zz}(z)\,f(\mathrm{z}_i)\,\hat{z}^2$
  - works for all orders

- Show later in the application than the missing terms can affect conclusions details

LoM of Ω

$$
\begin{aligned}
\frac{d}{d\theta'}\hat{\Omega}_{t+1}\langle a',\theta'\rangle &= -\int \overbrace{\underbrace{\delta\left(\bar{a}(a,\theta)-a'\right)}_{\iota_Z(\bar{a}-a')}\underbrace{\mu(\theta^{'}-\rho_\theta\theta)}_{\iota_{\theta'}\left(\theta'-\rho_\theta\theta\right)}}^{\overline{\Lambda}(a^{'},\theta^{'},a,\theta)}\underbrace{\bar{a}_{Z,t}(a,\theta)}_{\bar{a}_Z\cdot\hat{Z}_t}\,d\Omega^*\langle a,\theta\rangle \\
&\quad+\int \iota\left(\bar{a}(a,\theta)\leq a'\right)\mu(\theta^{'}-\rho_\theta\theta)d\hat{\Omega}_t\langle a,\theta\rangle. \\
&= -\left(\mathcal{M}\cdot\bar{a}_{Z,t}\right)\langle a',\theta'\rangle \\
&\quad+\int \overbrace{\delta\left(\bar{a}(a,\theta)-a'\right)\mu(\theta^{'}-\rho_\theta\theta)}^{\overline{\Lambda}(a^{'},\theta^{'},a,\theta)}\,\bar{a}_a(a,\theta)\frac{d}{d\theta}\hat{\Omega}_t\langle a,\theta\rangle da d\theta \\
&= -\left(\mathcal{M}\cdot\bar{a}_{Z,t}\right)\langle a',\theta'\rangle+\left(\mathcal{L}^{(a)}\cdot\frac{d}{d\theta}\hat{\Omega}_t\right)\langle a',\theta'\rangle
\end{aligned}
$$