Efficiency, Insurance, and Redistribution Effects of Government Policies*

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January 2023

Abstract

We decompose welfare effects of switching from government policy A to policy B into three components: gains in *aggregate efficiency* from changes in total resources; gains in *redistribution* from altered consumption shares that *ex-ante* heterogeneous households can expect to receive; and gains in *insurance* from changes in households' consumption risks. Our decomposition applies to a broad class of multi-person, multi-good, multi-period economies with diverse specifications of preferences, shocks, and sources of heterogeneity. It has several desirable properties that other decompositions lack. We apply our decomposition to two fiscal policy reforms in quantitative incomplete markets settings.

*We thank Casey Mulligan, Eduardo Davila, Harald Uhlig, Nicolas Werquin, Axelle Ferriere and participants at the 2022 Annual Meeting for the Society for Economic Dynamics for discussions and comments, and Leo Aparisi De Lannoy, Jiawei Fan and Karan Jain for excellent research assistance. Bhandari, Evans, and Golosov thank the NSF for support (grant #36354.00.00.00).

1 Introduction

We want to understand sources of differences in social welfare associated with alternative government policies. When households are heterogeneous, welfare depends on how efficiently goods and services are produced as well as how they are allocated across households. Welfare changes from reallocations are influenced by Pareto weights, so it is natural to want to isolate a contribution from redistribution across households.

We impute the welfare change from moving from policy A to policy B to three components that we call *aggregate efficiency*, *redistribution*, and *insurance*. Aggregate efficiency captures consequences of changes in aggregate resources. Redistribution captures changes in *ex-ante* consumption shares. Insurance captures changes in *ex-post* consumption risk.

Our decomposition has several desirable properties. It can be applied to static and dynamic stochastic economies with multiple goods and with preferences over goods that can vary across households. It isolates three components of welfare changes in intuitive ways. For instance, holding fixed both *ex-ante* expected consumption shares and *ex-post* risks in consumption for each household, welfare gains that arise from changes in aggregate resources are assigned entirely to the aggregate efficiency component. Holding fixed both the level of aggregate resources and consumption risk for each household, welfare changes that arise from changes in *ex-ante* consumption shares are assigned purely to the redistribution component. Similarly, welfare changes from a policy that reduces *ex-post* consumption risk but affects neither total resources nor expected consumption share of any household are assigned purely to the insurance component. Our decomposition is *reflexive* in the sense that each component of the welfare change from policy A to policy B is equal in magnitude and of the opposite sign to its counterpart for moving from policy B to policy A. We show that all decompositions that satisfy these properties agree with ours up to an appropriate approximation error.

We compare our decomposition to alternatives developed by Benabou (2002) and Floden (2001).[1] Benabou (2002) was the first to develop a decomposition of welfare effects of policy changes into the three components in a context of a specific economy, in which households have logarithmic preferences, and the innovations to consumption are always log-normal. Benabou's approach involves computing a certainty equivalent consumption for each household and then averaging it across households to construct a measure of total societal risk.

[1]Examples of papers that use these, or closely related decompositions, include Abbott et al. (2019), Cho et al. (2015), Conesa et al. (2009), Dyrda and Pedroni (2021), Guvenen et al. (2019), Heathcote et al. (2017), Koehne and Kuhn (2015), Nakajima and Takahashi (2020), Seshadri and Yuki (2004).

Benabou showed that in his economy this approach leads to a natural decomposition of welfare effects from policy changes into efficiency, insurance and redistribution components. Floden (2001) extended Benabou's approach to a general class of economies. Our decomposition agrees with the Benabou-Floden decomposition in Benabou's economy. Outside of that special case, we found that the approaches based on aggregating certainty-equivalent consumptions do not generally satisfy the abovementioned properties because they implicitly use inconsistent weights to aggregate household's gains from efficiency, insurance and redistribution into social welfare. As a result, such approaches may attribute an arbitrarily large or small share of welfare gain to the insurance component from a policy that reduces ex-post risk of households without affecting aggregate resources or their distributions among agents.

Subsequent to an earlier draft of our paper, Davila and Schaab (2022) proposed a decomposition of marginal policy changes. We show that their decomposition is subject to the same concerns as the decompositions that are based on aggregating certainty equivalents. Moreover, it differs from Benabou, Floden, and our decomposition even in cases when the latter decompositions agree, for instance, in the Benabou's economy. Similarly, Benabou, Floden, and our decomposition would fully attribute welfare gains from balanced growth – in which all households' consumptions grow at a common constant rate – to improvements in the aggregate efficiency component, while the decomposition of Davila and Schaab would interpret it as some mixture of all three components.

We apply our decomposition to two fiscal policy reforms in incomplete markets settings that are widely used in the macro literature. The first is a change in the income tax like that studied by Aiyagari (1995). We use this application to illustrate several insights that our decomposition brings, study its accuracy, and compare implications to those that would emerge from alternative decompositions. The second application is an increase in public debt in a setting with low interest rates in the spirit of Blanchard (2019). In our example that is calibrated to capture time-variation in debt servicing costs, issuing more public debt leads to a welfare loss even in presence of potential welfare gains from insurance.[2]

The rest of this paper is organized as follows. Section 2 describes the environment. Section 3 develops our decomposition and provides examples of applications of our decomposition. Section 4 compares our decomposition to those of Benabou, Floden, and Davila and Schaab. Section 5 applies our decomposition to two fiscal reforms in calibrated economies.

[2]Other applications of our decomposition can be found in Ferriere et al. (2022) and Beraja and Zorzi (2022).

Section 6 concludes. Proofs and technical details appear in the Appendix.

2 Environment

A unit measure of *ex-ante* heterogeneous households are subject to risk *ex-post*. Households are distributed on a $[0,1]$ interval endowed with a Lebesgue measure. Household i derives utility $U_i\left(\{c_{k,i}\}_k\right)$ from a bundle $\{c_{k,i}\}_k$ of goods. The bundle can be either finite or infinite. Each $c_{k,i} > 0$ is stochastic and drawn from a distribution that can differ across households. Expected utility of household i is $\mathbb{E}_i U_i\left(\{c_{k,i}\}_k\right)$, where $\mathbb{E}_i$ is a mathematical expectation with respect to household i's probability distribution over $\{c_{k,i}\}_k$. We assume that U_i is twice continuously differentiable and denote its first and second derivatives by $U_{k,i}, U_{km,i}$ for goods k, m. We assume that the joint distribution of stochastic processes $\{c_{k,i}\}_{i,k}$ has finite second moments and that expected utilities are well-defined. We use a shorthand $\mathbb{E}$ to denote an average over households. Thus, $\mathbb{E}x_i$ denotes $\int_{[0,1]} \mathbb{E}_i x_i di$ for any random variable x_i.

An *allocation* under a government *policy* is a collection of stochastic processes $\{c_{k,i}\}_{k,i}$ that assigns consumptions of all goods to all households. Welfare of an allocation, denoted by $\mathcal{W}$, is evaluated with Pareto weights $\{\alpha_i\}_i$ that satisfy $\alpha_i \geq 0$ and $\mathbb{E}\alpha_i = 1$, and is given by

$$\mathcal{W} \equiv \mathbb{E}\alpha_i U_i\left(\{c_{k,i}\}_k\right). \tag{1}$$

Equation (1) is commonly used in applied work to measure welfare and to evaluate counterfactual policies. Switching from government policy A to policy B alters the allocation and leads to welfare change $\mathcal{W}^B - \mathcal{W}^A$. We use superscripts $j \in \{A, B\}$ to denote variables under alternative policies. Our goal is to decompose a welfare change $\mathcal{W}^B - \mathcal{W}^A$ into economically interpretable components.

3 A decomposition

Subsection 3.1 analyzes a single good economy. Subsection 3.2 extends our analysis to settings with multiple goods.

3.1 Single good economy

When there is a single good, subscripts k are redundant so we can use U_c, U_{cc} to denote first and second derivatives of the utility function. A household's consumption satisfies the identity

$$c_i = \mathbb{E}c_i \times \frac{\mathbb{E}_i c_i}{\mathbb{E}c_i} \times \frac{c_i}{\mathbb{E}_i c_i} \equiv C \times w_i \times (1+\varepsilon_i)\,. \tag{2}$$

Aggregate consumption C measures the size of an aggregate "pie." Fraction w_i is the share of that pie that household i expects to receive. Shocks ε_i capture the uncertainty that household i faces. By construction, $\mathbb{E}_i\varepsilon_i = 0$ and $\mathbb{E}w_i = 1$.

Identity (2) motivates us to decompose a welfare change across two consumption allocations into components that measure *aggregate efficiency, redistribution,* and *insurance.* Before presenting our decomposition, it is useful to prescribe some desirable properties.

Property a. A welfare change from a policy that affects aggregate consumption C but not $\{w_i, \varepsilon_i\}_i$ should be imputed solely to *aggregate efficiency*;

Property b. A welfare change from a policy that affects expected shares $\{w_i\}_i$ but not C or $\{\varepsilon_i\}_i$ should be imputed solely to *redistribution*;

Property c. A welfare change from a policy that affects the stochastic process for $\{\varepsilon_i\}_i$ but not C or $\{w_i\}_i$ should be imputed solely to *insurance.*

Our notion of aggregate efficiency requires that redistribution and insurance are unaffected if every household's consumption is multiplied by the same positive scalar. It implies that all welfare gains from comparing allocations at two dates along a balanced growth path, when consumptions of all households grow at a common constant rate at all dates and states, are fully attributed to improvements in efficiency. This seems consistent with common usages of the term "efficiency." More generally, this notion also implies that redistribution and insurance remain unchanged so long as the distribution of expected consumption shares, $\{w_i\}_i$ and the distribution of consumptions relative to their means, $\{c_i/\mathbb{E}_i c_i\}_i$ are both unchanged. Once we accept Property (a) of pure aggregate efficiency, the other two properties follow naturally. In *redistribution*, we capture effects from reshuffling resources across households, that is, from changing expected consumption shares $\{w_i\}_i$, while keeping C and $\{\varepsilon_i\}_i$ fixed. In *insurance*, we capture changes in consumption risk that each household faces. Thus, we attribute to the insurance component the consequences of a policy-induced change in the distribution of $\{\varepsilon_i\}_i$ that keeps aggregate resources and expected consumption shares constant (i.e., pure mean-preserving spreads in consumption).

In addition, we want a decomposition not to depend on whether one compares policy B to policy A, or policy A to policy B. We call this property *reflexivity* and formally state it as follows.

Property d. The absolute value of each component of the welfare change from policy A to policy B equals its counterpart in moving from policy B to policy A.

Our approach rests on Taylor expansions of welfare difference $\mathcal{W}^B - \mathcal{W}^A$. Welfare $\mathcal{W}^j$ for $j \in \{A, B\}$ can be represented as a mapping from an allocation— i.e., from sequences and stochastic processes $\left\{C^j, w_i^j, \varepsilon_i^j\right\}_i$—to a real number. We expand $\mathcal{W}^j$ around a non-stochastic "midpoint" $\left\{C^Z, w_i^Z, \mathbf{0}\right\}_i$ defined as

$$C^Z \equiv \sqrt{C^A C^B},\ w_i^Z \equiv \sqrt{w_i^A w_i^B},\ c_i^Z \equiv C^Z w_i^Z. \tag{3}$$

Define quasi-weights $\phi_i \equiv \alpha_i U_{c,i}\left(c_i^Z\right) c_i^Z$, and coefficients of relative risk aversion $\gamma_i \equiv -U_{cc}\left(c_i^Z\right) c_i^Z / U_c\left(c_i^Z\right)$, and define components Γ, $\{\Delta_i, \Lambda_i\}_i$ as follows:

$$\Gamma \equiv \ln C^B - \ln C^A,\ \Delta_i \equiv \ln w_i^B - \ln w_i^A,\ \Lambda_i \equiv -\frac{1}{2}\left[var_i\left(\ln c_i^B\right) - var_i\left(\ln c_i^A\right)\right]. \tag{4}$$

Applying Taylor's theorem, we show (in the appendix) that

$$\mathcal{W}^B - \mathcal{W}^A \simeq \underbrace{\mathbb{E}\phi_i\Gamma}_{\text{agg. efficiency}} + \underbrace{\mathbb{E}\phi_i\Delta_i}_{\text{redistribution}} + \underbrace{\mathbb{E}\phi_i\gamma_i\Lambda_i}_{\text{insurance}}, \tag{5}$$

where "$\simeq$ " denotes equality up to a third-order remainder term in a Taylor series expansion.

Equation (5) shows that up to third-order expansion terms, the welfare effect $\mathcal{W}^B - \mathcal{W}^A$ can be represented as a sum of three terms. The first term is proportional to the increase in aggregate resources, Γ. The second term is proportional to a quantity that captures changes in households' expected consumption shares, $\{\Delta_i\}_i$. The third term is proportional to changes in *ex-post* risk and is captured by $\{\Lambda_i\}_i$. Although we consider a second-order expansion, there are no interaction terms among Γ, $\{\Delta_i\}_i$ and $\{\varepsilon_i\}_i$. We conclude that a natural way to represent contributions of aggregate efficiency, redistribution, and insurance to $\mathcal{W}^B - \mathcal{W}^A$ is in terms of the proportions $\frac{\mathbb{E}\phi_i\Gamma}{\mathbb{E}\phi_i[\Gamma+\Delta_i+\gamma_i\Lambda_i]}$, $\frac{\mathbb{E}\phi_i\Delta_i}{\mathbb{E}\phi_i[\Gamma+\Delta_i+\gamma_i\Lambda_i]}$, and $\frac{\mathbb{E}\phi_i\gamma_i\Lambda_i}{\mathbb{E}\phi_i[\Gamma+\Delta_i+\gamma_i\Lambda_i]}$ respectively.[3] These terms must sum to 1:

$$1 = \frac{\mathbb{E}\phi_i\Gamma}{\mathbb{E}\phi_i\left[\Gamma + \Delta_i + \gamma_i\Lambda_i\right]} + \frac{\mathbb{E}\phi_i\Delta_i}{\mathbb{E}\phi_i\left[\Gamma + \Delta_i + \gamma_i\Lambda_i\right]} + \frac{\mathbb{E}\phi_i\gamma_i\Lambda_i}{\mathbb{E}\phi_i\left[\Gamma + \Delta_i + \gamma_i\Lambda_i\right]}. \tag{6}$$

Terms in decomposition (5) have natural interpretations. Start with the efficiency compo-

[3] In general, the remainder term in a Taylor series expansion (5) depends on higher-order interactions of Γ, $\{\Lambda_i\}_i$ and $\{\varepsilon_i\}_i$ and cannot be partitioned unambiguously across the three components. Defining the proportions as we do implicitly assumes that residuals are split across these components in the same proportions as the first and second-order expansion terms. See the appendix for details.

nent, which is the sum across households of the percentage change in aggregate consumption scaled by quasi-weights ϕ_i. To see why it takes this form, expand the term inside the expectation operator: $\underbrace{\alpha_i \times U_{c,i}^Z \times c_i^Z}_{\phi_i} \times \Gamma$. Here, Γ is the growth rate of aggregate consumption, $c_i^Z \Gamma$ is how much household i's consumption increases at the aggregate growth rate, $U_{c,i}^Z c_i^Z \Gamma$ converts the change in consumption for household i into a change in utils of household i, and $\alpha_i U_{c,i}^Z c_i^Z \Gamma$ converts that into social utils. Reflexivity requires us to measure these changes at the midpoint Z, which also turns out to be a natural point that minimizes the error terms in the approximation. The quasi-weights $\{\phi_i\}_i$ thus transform percentage changes to welfare change $\mathcal{W}^B - \mathcal{W}^A$, measured in utils.

The insurance component depends on the coefficients of the relative risk aversion $\{\gamma_i\}_i$ and changes in the variance of log consumption. It can be interpreted in terms of certainty equivalences. Denote the certainty equivalent level of consumption for each household i under policy j by $c_i^{ce,j}$. It satisfies

$$U\left(c_i^{ce,j}\right) = \mathbb{E}_i U\left(c_i^j\right). \tag{7}$$

A second-order approximation of this relationship yields $\ln\left(\frac{c_i^{ce,j}}{\mathbb{E}_i c_i^j}\right) \simeq -\frac{1}{2}\gamma_i var_i(\ln(c_i^j))$. From the definition of Λ_i, an alternative way of representing the insurance component is

$$\text{insurance} \simeq \mathbb{E}\phi_i \left(\ln\left(\frac{c_i^{ce,B}}{\mathbb{E}_i c_i^B}\right) - \ln\left(\frac{c_i^{ce,A}}{\mathbb{E}_i c_i^A}\right)\right). \tag{8}$$

As with the efficiency component, the quasi-weights ϕ_i convert changes in the fraction of consumption that households are willing to give up to remove all uncertainty into changes in total welfare.

Finally, the redistribution component depends on changes in expected consumption shares $\{w_i\}_i$, captured by $\{\Delta_i\}_i$. To understand this term, it is instructive to write it as

$$\mathbb{E}\phi_i\Delta_i = \sqrt{C^A C^B} \int_{[0,1]} \alpha_i U_{c,i}^Z \ln \frac{w_i^B}{w_i^A} \sqrt{w_i^A w_i^B} di.$$

The integral in this expression resembles but differs from a Kullback–Leibler (K-L) statistical divergence, that would take the form $\int \ln \frac{w_i^B}{w_i^A} w_i^A di$ in our context. Like the K-L divergence, our redistribution component quantifies differences between distributions $\left\{w_i^A\right\}_i$ and $\left\{w_i^B\right\}_i$. (Note that $\left\{w_i^A\right\}_i$ and $\left\{w^B\right\}_i$ are both positive and sum to one, so that they are probability distributions.) There are two important differences from the K-L divergence. First, the

K-L divergence is not reflexive and violates our desired Property (d). By using the midpoint between the two distributions, $\sqrt{w_i^A w_i^B}$, rather than w_i^A, we overcome this problem. Second, the K-L divergence "weighs" resources given to each household i equally. That corresponds to a very specific point on the Pareto frontier. At an arbitrary point of the Pareto frontier, these weights are given by $\alpha_i U_{c,i}$.

We wrote the redistributive component as $\mathbb{E}\phi_i \Delta_i$ in formula (5) to emphasize a common structure underlying the three components and to illuminate its relationship to standard measures of divergence. An alternative way of representing the redistribution component is

$$\text{redistribution} \simeq \sqrt{C^A C^B} \mathbb{E} \alpha_i U_{c,i}^Z \left(w_i^B - w_i^A\right). \tag{9}$$

From (9) one can immediately see that redistribution measures changes in shares $w_i^B - w_i^A$ weighted with $\alpha_i U_{c,i}$. Evidently, the redistributive component is always zero if the planner is utilitarian and households have linear utility of consumption, consistent with the observation that in such settings the planner does not value redistribution

It is easy to verify that decomposition (6) satisfies Properties (a), (b), (c), and (d). A policy change that affects available aggregate resources C but not $\{w_i, \varepsilon_i\}_i$ implies that $\Delta_i = \Lambda_i = 0$ for all i; therefore the aggregate efficiency component of such a policy is 1. Similar arguments verify that Properties (b) and (c) are satisfied. Reflexivity Property (d) can be verified by noticing that moving from policy B to policy A implies that both the numerator and the denominator of each fraction changes only its sign, leaving ratios unchanged.

The statement that decomposition (5) satisfies Properties (a), (b), (c), and (d) naturally raises the question if there are other decompositions that do the same. In the appendix, we show that any additive decomposition that satisfies Properties (a), (b), (c), and (d) agrees with decomposition (5) up to a third-order error. To see the intuition for this claim, consider an alternative decomposition given by adding and subtracting $a\Gamma\Delta_i$ to (5):

$$\mathcal{W}^B - \mathcal{W}^A \simeq \underbrace{\mathbb{E}\phi_i\Gamma + a\Gamma\Delta_i}_{\text{agg. efficiency}} + \underbrace{\mathbb{E}\phi_i\Delta_i - a\Gamma\Delta_i}_{\text{redistribution}} + \underbrace{\mathbb{E}\phi_i\gamma_i\Lambda_i}_{\text{insurance}}.$$

This decomposition satisfies properties (a), (b), and (c) but lacks property (d) because going from policy B to policy A will not switch the sign of the term $a\Gamma\Delta_i$. In the appendix, we show that up the order of error in (5), we need only to be concerned about interaction terms between $\{\Gamma, \Delta_i, \Lambda_i\}$ and that requiring the decomposition to be reflexive will impose a restriction that those terms equal zero.

3.1.1 Order of approximation and an alternative decomposition

How large is the omitted third-order residual in equation (5) and how does its size depend on the difference between allocations under policies A and B? We provide a short discussion of these questions, leaving detailed proofs for the appendix.

Define a space of sequences and stochastic processes $\left\{\tilde{\Gamma}, \tilde{\Delta}_i, \tilde{\varepsilon}_i\right\}_i$ endowed with an appropriate norm $||\cdot||$. From (2), any allocation $\{c_i^j\}_i$ can be mapped to a point in that space. We use $||\Gamma, \Delta, \varepsilon^B - \varepsilon^A||$ to measure the distance between allocations under two policies, $\left\{c_i^A\right\}_i$ and $\left\{c_i^B\right\}_i$ and denote it $||c^B - c^A||$. While the residual in equation (5) goes to zero as $||c^B - c^A|| \to 0$, little can be said theoretically about the relative speeds at which the residual and $||c^B - c^A||$ converge to zero. This is because when we expand around a non-stochastic point $\left\{c_i^Z\right\}_i$, processes $\left\{c_i^B\right\}_i$ and $\left\{c_i^A\right\}_i$ need not converge to this point as $||c^B - c^A|| \to 0$.

We can modify our decomposition to ensure that the approximation error shrinks to zero at a rate faster than $||c^B - c^A||^2$. Write the stochastic process c_i as an explicit mapping from a vector of primitive shocks ξ into consumption $c_i(\xi)$, with the distribution of ξ given by $\Pr_i(d\xi)$. Equation (2) can be rewritten as $c_i(\xi) = C \times w_i \times \epsilon_i(\xi)$ where $\epsilon_i(\xi) \equiv c_i(\xi) / \mathbb{E}_i c_i$. Let the expansion point be $\tilde{c}_i^Z(\xi) \equiv \sqrt{C^A C^B} \times \sqrt{w_i^A w_i^B} \times \sqrt{\epsilon_i^A(\xi)\, \epsilon_i^B(\xi)}$ and let $\delta_i(\xi) \equiv \ln \epsilon_i^B(\xi) - \ln \epsilon_i^A(\xi)$, and $\tilde{\phi}_i(\xi) \equiv \alpha_i U_{c,i}\left(\tilde{c}_i^Z(\xi)\right) \tilde{c}_i^Z(\xi)$. Then the welfare decomposition can be written

$$\mathcal{W}^B - \mathcal{W}^A = \underbrace{\mathbb{E}\tilde{\phi}_i \Gamma}_{\text{agg. efficiency}} + \underbrace{\mathbb{E}\tilde{\phi}_i \Delta_i}_{\text{redistribution}} + \underbrace{\mathbb{E}\tilde{\phi}_i \delta_i}_{\text{insurance}} + o\left(\|\Gamma, \Delta, \delta\|^2\right). \tag{10}$$

This decomposition retains the four properties of decomposition (5) discussed in Section 3.1 and guarantees that the reminder term converges to zero at a faster rate than $\left\|c^B - c^A\right\|^2$.

Decomposition (10) lends itself to an *exact* decomposition when the size of the reform is small. For instance, let τ be a parameter so that household i's allocation is given by

$$c_i(\xi, \tau) = \mathbb{E}c_i(\xi, \tau) \times \frac{\mathbb{E}_i c_i(\xi, \tau)}{\mathbb{E}c_i(\xi, \tau)} \times \frac{c_i(\xi, \tau)}{\mathbb{E}_i c_i(\xi, \tau)} \equiv C(\tau) \times w_i(\tau) \times (1 + \varepsilon_i(\xi, \tau)). \tag{11}$$

Define $\hat{\phi}_i(\tau, \xi) \equiv \alpha_i U_{c,i}(c_i(\xi, \tau))\, c_i(\xi, \tau)$, and $\partial_\tau \mathcal{W}(\tau) \equiv \lim_{\|h\| \to 0} \frac{\mathcal{W}^{\tau+h} - \mathcal{W}^\tau}{\|h\|}$ be the welfare gain from a small change in τ. A marginal version of (10) is given by

$$\partial_\tau \mathcal{W}(\tau) = \underbrace{\mathbb{E}\hat{\phi}_i(\tau)\,\partial_\tau \ln \mathbb{E}c_i(\tau)}_{\text{agg. efficiency}} + \underbrace{\mathbb{E}\hat{\phi}_i(\tau)\,\partial_\tau \ln\left(\frac{\mathbb{E}_i c_i(\tau)}{\mathbb{E}c_i(\tau)}\right)}_{\text{redistribution}}$$
$$+ \underbrace{\mathbb{E}cov\left[\hat{\phi}_i(\tau), \partial_\tau \ln\left(c_i(\tau)/\mathbb{E}_i c_i(\tau)\right)\right]}_{\text{insurance}}, \tag{12}$$

where *cov* is a cross-sectional covariance. Its easy to verify that decomposition (12) satisfies properties (a)–(d).

While decomposition (10) has nicer theoretical properties as the difference between allocations induced by policies A and B becomes small, we did not find any meaningful differences between decompositions (10) and (5) for all the examples that we considered. For this reason, for the rest of this paper, we focus on decomposition (5), both because it is easier to compute and because $\{\Lambda_i\}_i$ maps directly into statistics that are routinely reported in implementations of quantitative models.

3.2 Decomposition for a multi-good economy

It is straightforward to extend our decomposition to multi-good settings. To decompose welfare gains into components, first compute points of approximation $\left\{c_{k,i}^Z\right\}_k$ for each good as in equation (3). Then extend definitions of Γ_k and $\Delta_{k,i}$ from equation (4) to every good k and define $\Lambda_{km,i}$ for each pair of goods k, m as

$$\Lambda_{km,i} \equiv -\frac{1}{2}\left[cov_i\left(\ln c_{k,i}^B, \ln c_{m,i}^B\right) - cov_i\left(\ln c_{k,i}^A, \ln c_{m,i}^A\right)\right].$$

Let $U_{k,i}, U_{km,i}$ be first and second derivatives of U_i evaluated at $\left\{c_{k,i}^Z\right\}_k$ and let weights $\{\phi_{k,i}\}_k$ and cross-elasticities $\{\gamma_{km,i}\}_{k,m}$ be defined as

$$\phi_{k,i} \equiv \alpha_i U_{k,i} c_i^Z, \quad \gamma_{km,i} \equiv -\frac{U_{km,i} c_{m,i}^Z}{U_{k,i}}.$$

Using the same steps as in the one good economy from Section 3.1, we can show that

$$\mathcal{W}^B - \mathcal{W}^A \simeq \text{agg. efficiency} + \text{redistribution} + \text{insurance}, \tag{13}$$

where

$$\begin{aligned}
\text{agg. efficiency} &= \mathbb{E}\sum_k \phi_{k,i}\Gamma_k, \\
\text{redistribution} &= \mathbb{E}\sum_k \phi_{k,i}\Delta_{k,i}, \\
\text{insurance} &= \mathbb{E}\sum_k\sum_m \phi_{k,i}\gamma_{km,i}\Lambda_{km,i}.
\end{aligned}$$

When utility is separable across all goods, this decomposition just computes decomposition (5) for each good and then sums each component across all goods. When utility is not separable, proper accounting for the insurance component requires adding changes in the covariances in dispersions of different goods weighted with cross-elasticities $\gamma_{km,i}$.

This approach applies directly to decomposing welfare gains from switching from policy A to policy B in infinite horizon economies. A typical application in quantitative macro is to find an invariant distribution under some policy τ^A, use that as an initial condition for an economy in which a different policy τ^B is introduced. One then compares welfare in the invariant distribution under policy τ^A to welfare along the transition path under policy τ^B (see Guerrieri and Lorenzoni (2017), Rohrs and Winter (2017) or Section 5 of this paper for examples). The utility function is typically assumed to be of the form

$$\mathbb{E}_i U_i\left(\{c_{k,i}\}_k\right) = \mathbb{E}_i \sum_{k=1}^{\infty} \beta^{k-1} u\left(c_{k,i}\right),$$

where β is the discount factor, u is the within-period utility function, k is time and $\mathbb{E}_i$ is the period 0 conditional expectation.

These applications map directly into the framework developed in this section. Each good k in our multi-good setup corresponds to consumption in period k, tuple $\left(C_k^j, w_{k,i}^j, \varepsilon_{k,i}^j\right)$ consists of aggregate consumption, consumption shares of household i, and consumption risk for household i in period k under policies $j \in \{A, B\}$. If the initial distribution is the invariant distribution under policy A, then sequence $\left\{C_k^A\right\}_k$ takes the same values for all k, but sequences $\left\{w_{k,i}^A, \varepsilon_{k,i}^A\right\}_k$ can change over time due, for example, to idiosyncratic shocks experiences by households.

3.3 Comparing invariant distributions

In some applications, it can be difficult to compute a full transition path induced by changing government policies, so researchers often rely on comparing welfare across steady states.

That is, they compare welfare under the invariant distribution arising from policy B to welfare under the invariant distribution arising from policy A. We provide a way to extend our framework to such settings.

Suppose that an invariant distribution under policy j is characterized by the probability measure, μ^j, on some compact space of household characteristics $S \subset \mathbb{R}^n$. In many applications (e.g., Aiyagari (1995), Floden (2001), Conesa et al. (2009)) welfare is defined as an integral over households' expected utilities,

$$\mathcal{W}^j = \int_S \mathbb{E}_s U\left(\left\{c^j_{k,s}\right\}_k\right) d\mu^j(s),$$

where $\mathbb{E}_s$ is the expected utility of household $s \in S$. In order to apply our approach to decompose the welfare change from A to B, we need households to be living in the same probability space; they might not if $\mu^A \neq \mu^B$. Our solution is to work in the space of percentiles: we map the i^{th} percentile under policy A to the i^{th} percentile under policy B.

It is easiest to consider first a case in which the space of characteristics is unidimensional so that $n = 1$, and μ^j is described by a probability density, $f^j : S \mapsto \mathbb{R}$, with $f^j(s) > 0$ for all $s \in S$. Let F^j be the cdf of f^j. For any c^j_s that is function of $s \in S$, define $\bar{c}^j_i$ by $\bar{c}^j_{F(s)} = c^j_s$. By construction, $\bar{c}^j_i$ is uniquely defined for $i \in [0,1]$. Standard "integration by substitution" arguments (see, e.g. Corollary 3.7.2 in Bogachev (2007)) imply that

$$\mathcal{W}^j = \int_S \mathbb{E}_s U\left(\left\{c^j_{k,s}\right\}_k\right) d\mu^j(s) = \int_S \mathbb{E}_s U\left(\left\{c^j_{k,s}\right\}_k\right) f^j(s)\, ds = \int_{[0,1]} \mathbb{E}_i U\left(\left\{\bar{c}^j_{k,i}\right\}_k\right) di.$$

The last term is a special case of our definition of welfare (1), so all the steps from Section 3.2 apply directly.

This approach generalizes to any n via induction. We verify this for $n = 2$. Suppose that $S = \mathcal{A} \times \Theta \subset \mathbb{R}^2$ and assume probability density $f^j(a,\theta)$ under policy j. Let $f^j(\theta) \equiv \int_{\mathcal{A}} f^j(a,\theta)\, da$. We can write welfare as

$$\mathcal{W}^j = \int_\Theta \left[\int_{\mathcal{A}} \mathbb{E}_{(a,\theta)} U\left(\left\{c^j_{k,(a,\theta)}\right\}_k\right) \frac{f^j(a,\theta)}{f^j(\theta)} da\right] f^j(\theta)\, d\theta.$$

Applying the same procedure that we described for $n = 1$ case twice, first to the inner integral, and then to the outer one, we can represent welfare

$$\mathcal{W}^j = \int_{[0,1]^2} \mathbb{E}_{(i,\iota)} U\left(\left\{\bar{c}^j_{k,(i,\iota)}\right\}_k\right) di d\iota. \tag{14}$$

Application of the Section 3.2 decomposition is now straightforward, except that now households are distributed over $[0,1]^2$ rather than $[0,1]$. An application of the $n=2$ procedure is studied in Section 5.1.

4 Alternative decompositions

Motivated by the same questions that concern us, Benabou (2002), Floden (2001), and more recently, Davila and Schaab (2022) decomposed policy-induced welfare changes into counterparts of our efficiency, insurance, and redistribution components. In this section, we use a stylized economy to compare our decomposition with these alternatives. We start with the decompositions of Benabou (2002) and Floden (2001) that are based on certainty equivalents and widely used in quantitative work.

4.1 Benabou (2002) and Floden (2001) decompositions

Benabou (2002) was the first paper to develop a systematic decomposition of the welfare effects of policy changes. Benabou did it in the context of a specific economy that is described in Section 4.2. Floden (2001) proposed an extension of Benabou's approach to a wider class of economies.

It is easiest to describe their ideas in the single good environment. Both decompositions require that all households have identical preferences U and start by computing a certainty equivalent level of consumption for each household i under policy j, $c_i^{ce,j}$, as in equation (7). Then they define the aggregate certainty equivalent $C^{ce,j} \equiv \mathbb{E}c_i^{ce,j}$. As in Section 3, we use $C^j = \mathbb{E}c_i^j$ to denote aggregate consumption under policy j.

In Benabou's economy, preferences over consumption are logarithmic, $U(c) = \ln c$, and he considered the following decomposition of the effect of welfare changes from policy A to another policy B:

$$\begin{aligned}\mathcal{W}^B - \mathcal{W}^A &= \left[\underbrace{\ln C^B - \ln C^A}_{\text{agg. efficiency}}\right] + \left[\underbrace{\left\{\mathcal{W}^B - \mathcal{W}^A\right\} - \left\{\ln C^{ce,B} - \ln C^{ce,A}\right\}}_{\text{redistribution}}\right] \\ &\quad + \left[\underbrace{\left\{\ln C^{ce,B} - \ln C^{ce,A}\right\} - \left\{\ln C^B - \ln C^A\right\}}_{\text{insurance}}\right]. \qquad (15)\end{aligned}$$

Benabou shows that in his economy the three terms in square brackets have natural interpretations of aggregate efficiency, redistribution, and insurance, respectively.[4]

Floden (2001) proposed to extend Benabou's approach to general settings. As a first step, he computes numbers p^j_{insur} and p^j_{redis} using

$$U\left(\left(1-p^j_{insur}\right)C^j\right)=U\left(C^{ce,j}\right),\quad U\left(\left(1-p^j_{redis}\right)C^{ce,j}\right)=\mathbb{E}\alpha_i U\left(c_i^{ce,j}\right). \tag{16}$$

Then he constructs contributions of aggregate efficiency, redistribution, and insurance as

$$1+\omega_{eff}\equiv\frac{C^B}{C^A},\ 1+\omega_{redis}\equiv\frac{1-p^B_{redis}}{1-p^A_{redis}},\ 1+\omega_{insur}\equiv\frac{1-p^B_{insur}}{1-p^A_{insur}}.$$

Similarly, the total welfare change from policy A to policy B is computed in consumption units as

$$\mathbb{E}\alpha_i U\left(c_i^B\right)=\mathbb{E}\alpha_i U\left((1+\omega)\,c_i^A\right). \tag{17}$$

Floden shows that when U has the CRRA form $U(c)=\frac{c^{1-\gamma}}{1-\gamma}$, with $\gamma>0$, $\gamma\neq 1$ and $U(c)=\ln c$ for $\gamma=1$, then the following relationship holds

$$\ln(1+\omega)=\ln(1+\omega_{eff})+\ln(1+\omega_{redis})+\ln(1+\omega_{insur}), \tag{18}$$

with the three terms corresponding to our notions of aggregate efficiency, redistribution, and insurance, respectively. Note that it coincides with Benabou's decomposition when $\gamma=1$.

4.2 An illustration of decompositions in the stylized Benabou's economy

In this section, we use a stylized version of the economy considered by Benabou to illustrate the workings of these decompositions. Benabou's economy allows rich heterogeneity, yet it is tractable and yields intuitive closed-form expressions for key equilibrium objects. This economy and its extensions have been widely used in the literature (see, e.g., Heathcote et al. (2017), Heathcote et al. (2020)).

Consider a static economy. Households have preferences over stochastic consumption good c_i and labor l_i represented by

$$\mathbb{E}_i\left[\ln c_i-\frac{1}{1+\eta}l_i^{1+\eta}\right].$$

[4] Benabou and Floden use slightly different terminologies in naming their decomposition terms, both from each other and from our paper. To avoid confusion, we use our terminology throughout.

Pre-tax earnings of household i are $y_i = \theta_i l_i$, where θ_i is labor productivity that satisfies $\theta_i = \exp(e_i + \xi_i)$. The first component of productivity $e_i \sim \mathcal{N}\left(-\frac{\upsilon_e^2}{2}, \upsilon_e^2\right)$ captures heterogeneity in initial, ex-ante skill endowments across households. The second component $\xi_i \sim \mathcal{N}\left(-\frac{\upsilon_\xi^2}{2}, \upsilon_\xi^2\right)$ represents idiosyncratic productivity shocks that are realized ex-post. After-tax labor income is $\bar{\tau} y_i^{1-\tau}$, where τ is the degree of tax progressivity and $\bar{\tau}$ is a function of τ such that the net tax revenues are zero. Households hold no assets and consume their after-tax labor income each period. Welfare $\mathcal{W} = \mathbb{E}\alpha_i \left[\ln c_i - \frac{1}{1+\eta} l_i^{1+\eta}\right]$, where Pareto weights α_i may depend on ex-ante heterogeneity e_i but not ex-post shocks ξ_i.[5]

Benabou focuses on understanding the effect of changes in the tax progressivity parameter τ. Logarithmic utility in consumption and the absence of non-labor income imply that all households choose the same labor supply in all periods, $l_i(e_i, \xi_i, \tau) = (1-\tau)^{\frac{1}{1+\eta}}$. This implies that aggregate labor $L(\tau)$ and aggregate consumption $C(\tau)$ satisfy

$$C(\tau) = L(\tau) = (1-\tau)^{\frac{1}{1+\eta}}.$$

It is easy to derive closed-form expressions for individual consumption and welfare:

$$c_i(e_i, \xi_i, \tau) = C(\tau) \times \exp\left((1-\tau) e_i + \tau(1-\tau)\frac{\upsilon_e^2}{2}\right) \times \exp\left((1-\tau)\xi_i + \tau(1-\tau)\frac{\upsilon_\xi^2}{2}\right), \tag{19}$$

and

$$\mathcal{W}(\tau) = \left[\ln C(\tau) - \frac{1}{1+\eta} L(\tau)^{1+\eta}\right] - \left[(1-\tau)^2 \frac{\upsilon_e^2}{2} - (1-\tau) cov(\alpha_i, e_i)\right] - \left[(1-\tau)^2 \frac{\upsilon_\xi^2}{2}\right]. \tag{20}$$

These expressions show that individual consumption and welfare can all be conveniently separated into three components with natural economic interpretation: the aggregate efficiency component, which depends on aggregate variables C and L; the redistribution component, which depends on the ex-ante heterogeneity e_i; and the insurance component, which depends on the ex-post risk ξ_i. An increase in tax progressivity affects all three components: it distorts labor supply and lowers aggregate consumption reducing efficiency; it decreases ex-ante differences in expected earnings providing redistribution; and it reduces the variance

[5] The stylized static economy is chosen to preserve all key features of Benabou's model in the simplest setting. Benabou (2002) considers a dynamic economy and assumes that each period households are subject to multiplicative random walk shocks drawn from a fixed log-normal distribution. It can be shown that each period t of his economy is equivalent to our static economy where the variance of ex-post shocks is given by $\upsilon_{\xi,t}^2$.

of ex-post earnings providing insurance.

We now apply Benabou and Floden decompositions (which coincide in this setting) to this economy and compare them to our decomposition. Individual and aggregate certainty equivalent consumption can be found in closed form and satisfy

$$c_i^{ce}\left(e_i,\tau\right)=C(\tau)\times\exp\left\{-\left((1-\tau)e_i+\tau(1-\tau)\frac{\nu_e^2}{2}\right)\right\}\times\exp\left\{-(1-\tau)^2\frac{\nu_\xi^2}{2}\right\},$$

$$C^{ce}\left(\tau\right)=C(\tau)\times\exp\left\{-(1-\tau)^2\frac{\nu_\xi^2}{2}\right\}.$$

Using definitions of Benabou's components from equation (15) and ours from equation (5) we get

$$\begin{aligned}\ln\left(1+\omega_{eff}\right)&=\frac{1}{1+\eta}\ln\left(\frac{1-\tau^B}{1-\tau^A}\right)\left[1-\frac{\tau^B-\tau^A}{\ln\left(\frac{1-\tau^B}{1-\tau^A}\right)}\right]\\&=\left(\frac{1}{1+\eta}\right)\left(\ln\frac{1-\tau^B}{1-\tau^A}\right)\left(1-\sqrt{\left(1-\tau^A\right)\left(1-\tau^B\right)}\right)+O\left(\left|\ln\frac{1-\tau^B}{1-\tau^A}\right|^3\right)\\&=\mathbb{E}\phi_i\Gamma+O\left(\left|\ln\frac{1-\tau^B}{1-\tau^A}\right|^3\right),\end{aligned}\tag{21}$$

$$\begin{aligned}\ln\left(1+\omega_{redis}\right)&=\frac{\upsilon_e^2}{2}\left[\left(1-\tau^A\right)^2-\left(1-\tau^B\right)^2\right]+cov\left(\alpha_i,e_i\right)\left[\tau^A-\tau^B\right]\\&=\mathbb{E}\phi_i\Delta_i,\end{aligned}\tag{22}$$

and

$$\begin{aligned}\ln\left(1+\omega_{insur}\right)&=\frac{\upsilon_\xi^2}{2}\left[\left(1-\tau^A\right)^2-\left(1-\tau^B\right)^2\right]\\&=\mathbb{E}\phi_i\gamma_i\Lambda_i.\end{aligned}\tag{23}$$

Thus, all three decompositions agree up to the third-order errors, with redistribution and insurance components coinciding exactly.[6] If we consider the effect of a marginal policy

[6] This third-order residual in the aggregate efficiency term emerges because we approximate the non-linear function $\frac{1}{1+\eta}\left[L\left(\tau^B\right)^{1+\eta}-L\left(\tau^A\right)^{1+\eta}\right]$ with a linear function $\left[\ln L\left(\tau^B\right)-\ln L\left(\tau^A\right)\right]$. The residual goes to zero at the rate $O\left(\left|\ln\frac{1-\tau^B}{1-\tau^A}\right|^3\right)$ as $\tau^B\rightarrow\tau^A$. To put this into perspective, Feenberg et al. (2017) estimate parameter τ for the U.S. for various years and find that between 1980 and 2010 it varied between 0.07 and

change, $\tau^B = \tau^A + d\tau$ in the limit as $d\tau \to 0$, previous expressions imply that all three decompositions agree exactly. We provide details in the appendix.

4.3 Decompositions in general economies

The result that ours and the Benabou-Floden decompositions agree in the Section 4.2 economy is driven by several special features. To see this, it is helpful to first apply our second-order approximation to the $\ln(1+\omega_{insur})$, the insurance component of Floden's decomposition. In the appendix, we show that

$$\ln(1+\omega_{insur}) \simeq \mathbb{E}\phi_i^{BF}\left(\ln\left(\frac{c_i^{ce,B}}{\mathbb{E}_i c_i^B}\right) - \ln\left(\frac{c_i^{ce,A}}{\mathbb{E}_i c_i^A}\right)\right), \tag{24}$$

where $\phi_i^{BF} = \frac{c_i^Z}{C^Z}$, which is the ex-ante consumption share of household i, or w_i^Z as in equation (3). Comparison to (8) reveals that one of the key differences between the two decompositions is the quasi-weights that are used to aggregate changes in individual certainty equivalents. Our quasi-weights $\{\phi_i\}_i$, with $\phi_i = \alpha_i U_{c,i}^Z c_i^Z$, incorporate both the households' marginal utilities of consumption and Pareto weights. This ensures that changes in households' risks are aggregated properly into households' utilities and social welfare respectively. In contrast, Benabou-Floden quasi-weights $\{\phi_i^{BF}\}_i$ depend only on the ex-ante consumption shares, and there is no a-priori reason to expect such weights to be consistent with the way a social welfare function aggregates utilities of households. Since the sum of the three components must equal the change in total welfare, this necessarily has repercussions for the other two components.

Thus, these two decompositions will generically disagree. They agree in Benabou's economy because it has a property that changes in the progressivity of the tax code induces the same change in the variance of log consumption, Λ_i, for all households. Under log utility, or $\gamma_i = 1$, this property renders changes in certainty equivalents equal across households,

$$\ln\left(\frac{c_i^{ce,B}}{\mathbb{E}_i c_i^B}\right) - \ln\left(\frac{c_i^{ce,A}}{\mathbb{E}_i c_i^A}\right) = \frac{\upsilon_\xi^2}{2}\left[\left(1-\tau^A\right)^2 - \left(1-\tau^B\right)^2\right].$$

Additionally, in Benabou's economy our quasi-weights simplify to $\phi_i = \alpha_i$ so both sets of quasi-weights aggregate to one, which implies, from equations (8) and (24), that both decompositions yield an identical contribution of the insurance component.

0.1. For tax policies in this range, $\left|\ln\frac{1-\tau^B}{1-\tau^A}\right|^3 \leq \left(\ln\frac{0.93}{0.9}\right)^3 \approx 0.000035.$

More generally, while there is a good economic rationale for using individual consumption equivalent c_i^{ce} to capture the effect of individual risk, there is no reason to expect that C^{ce} would be a good measure for societal risk.[7] The feature in Benabou's economy that changes in tax progressivity affects risk of all households (measured as volatility of the log consumption) equally makes it irrelevant how individual risks are aggregated. We next formalize the role of this property in "BF decompositions"[8] and explain problems that emerge when this property does not hold.

Lemma 1. *If there exists some* Λ *such that*

$$\frac{1}{2}\left(var_i\left(\ln c_i^A\right) - var_i\left(\ln c_i^B\right)\right) = \Lambda \text{ for all } i, \tag{25}$$

and the utility function U *is CRRA then the BF decomposition satisfies Properties (a), (b), and (c) and coincides with our decomposition, equation* (6), *up to a residual term that approaches zero as* $\left\|\Gamma^j, \Delta^j, \varepsilon^j\right\| \to 0$ *for* $j \in \{A, B\}$.

When condition (25) is not true, the BF decomposition (as well as any decomposition that strives to use C^{ce} as a measure of societal value of risk) would typically violate Properties (a), (b), and (c). Moreover, the consequences of the violation of those properties, and the resulting departure from an economically-meaningful decomposition of welfare changes, can be arbitrarily large. To illustrate this point, we go back to Benabou's economy presented in Section 4.2 but make three modifications to it: we (i) assume that the labor supply is inelastic, $\eta = \infty$; (ii) allow households to differ in the ex-post risk $\upsilon_{i,\xi}^2$; and (iii) abstract from taxes (set $\tau = 0$, policy$A$) but instead study the effect of introducing Arrow securities that are conditioned on the realization of the ex-post shocks ξ_i (policy B). The purpose of these modifications is simple. Assumptions (i) and (iii) shut down all channels of policy response

[7] A related observation is that certainty-equivalence approach is not well-defined if consumers' consumption bundles differ in more than one good, and preferences over those goods are not separable. In Floden (2001) economy, consumers are heterogeneous in both consumption and labor, and to compute individual certainty equivalent of consumption Floden experiments with setting individual labor as various fixed values.

[8] We refer to the decomposition

$$1 = \frac{\ln(1+\omega_{eff})}{\ln(1+\omega_{eff}) + \ln(1+\omega_{redis}) + \ln(1+\omega_{insur})} + \frac{\ln(1+\omega_{redis})}{\ln(1+\omega_{eff}) + \ln(1+\omega_{redis}) + \ln(1+\omega_{insur})} + \frac{\ln(1+\omega_{insur})}{\ln(1+\omega_{eff}) + \ln(1+\omega_{redis}) + \ln(1+\omega_{insur})}$$

as the "BF decomposition". Benabou's decomposition is a special case of it, but Benabou developed it only to study the effects of specific policy reforms in the economy he considered, and our decomposition aligns with his in that economy. The problems that we point out next apply not only to Floden's extension but also to any approach that may use the aggregate certainty equivalent consumption C^{ce} as a measure of societal risk in a general class of economies.

other than insurance. Under these assumptions, all households face pure endowment risk. Policy B provides means to insure that risk. In an equilibrium under policy B, households i buy Arrow securities to fully insure household-i specific ex-post risk. Thus, neither total resources nor their allocation across ex-ante heterogeneous households are effected, and all welfare gains from policy B come from insurance. Assumption (ii) allows us to consider both the case when condition (25) is satisfied ($\upsilon_{i,\xi}^2$ is the same for all i) and when it is not.

It is easy to verify that our decomposition (6) always attributes 100% of welfare gain to the insurance component. Closed-form expressions are not available for the BF decomposition, but for small risks, the corresponding insurance share is $\frac{\mathbb{E}\upsilon_{\xi,i}^2+cov(w_i^Z,\upsilon_{\xi,i}^2)}{\mathbb{E}\upsilon_{\xi,i}^2+cov(\alpha_i,\upsilon_{\xi,i}^2)}$. The expression reveals that the BF decomposition would attribute 100% of welfare gain to the insurance component only if risk $\upsilon_{\xi,i}^2$ is uncorrelated with ex-ante heterogeneity e_i (in particular, if $\upsilon_{\xi,i}^2$ is the same for all i). More generally, the BF decomposition can attribute anything from 0% to ∞% of the welfare gain to the insurance component (and anything from 100% to $-\infty$% to the redistribution component), depending on correlations between α_i, e_i and $\upsilon_{\xi,i}^2$.[9]

This failure of the BF decompositions to attribute 100% of the welfare gain to insurance arises because they aggregate individual insurance gains with quasi-weights that do not depend on the social weights of those households. While Pareto weights $\{\alpha_i\}_i$ appearing only in the redistribution component might seem like a desirable property, a momentary reflection reveals that such decompositions cannot lead to consistent aggregation of individual gains from insurance and redistribution. The BF decomposition (as would any composition that uses C^{ce} as a societal measure of risk) takes a stance on the implicit weights with which it aggregates individual insurance gains, which in general, is inconsistent with the weights that are used to aggregate gains from redistribution.

To illustrate the consequence of this, consider a special case of the previous example, with only two groups of households but with only one group facing ex-post risk, $0 = \upsilon_{1,\xi}^2 < \upsilon_{2,\xi}^2$. Suppose that the social welfare function assigns the Pareto weight of 0 to the second group.

[9] While this example is not particularly realistic as it was specifically constructed to illustrate our main point in the simplest way, the features it highlights are present in many economies widely used in quantitative macro applications. Consider, for example, a standard Aiyagari (1994) economy (along the lines we show in Section 5.1) or even Benabou's economy with non-labor income, such as assets (see the appendix for a worked out example). A proportional increase in taxes, under calibrated low values of the Frisch elasticity of labor supply, would produce roughly equal decrease in the variance in earnings for all households. But asset heterogeneity will imply that variance of consumption would be different for low-wealth household as compared to high-wealth households. In the models with a realistic distribution of wealth, that can lead to a large misattribution of the sources of welfare changes from tax policies. See Bhandari et al. (2021) for an illustration of this point in a calibrated HANK economy.

Policy B strictly improves the welfare of group 2 by providing it with better insurance but it has no effect on social welfare. Any welfare decomposition that does not use Pareto weights will find improvements in the insurance component of the welfare decomposition and, as a result, would need to adjust redistribution or efficiency components to ensure that their sum equals zero. In contrast, this problem does not arise in our decomposition since insurance gains of each household are weighted with their social weights.[10]

4.4 Comparison to Davila and Schaab decomposition

Subsequent to our work, Davila and Schaab (2022) propose a decomposition of welfare gains into similar labels. They develop their decomposition for marginal changes in policy, that is, welfare gains for changes when a policy τ is changed to $\tau + d\tau$, and take limits as $d\tau \to 0$. In our notation from Section 3.1.1, Davila and Schaab's decomposition can be written as

$$\begin{aligned}
\partial_\tau W^{DS} = & \underbrace{\int \Pr(d\xi)\left[\left(\int \frac{dc_i(\xi,\tau)}{d\tau}di\right) \times \left(\int \frac{u_i'(c_i(\xi,\tau))}{\int \Pr(d\xi)\, u_i'(c_i(\xi,\tau))}di\right)\right]}_{\text{DS-agg. efficiency}} \\
& + \underbrace{\int \Pr(d\xi)\, cov\left\{\frac{u_i'(c_i(\xi,\tau))}{\int \Pr(d\xi)\, u_i'(c_i(\xi,\tau))}, \frac{dc_i(\xi,\tau)}{d\tau}\right\}}_{\text{DS-insurance}} \\
& \underbrace{+ cov\left\{\frac{\int \alpha_i u_i'(c_i(\xi,\tau))\Pr(d\xi)}{\int\int \alpha_i u_i'(c_i(\xi,\tau))\, di \Pr(d\xi)}, \int \Pr(d\xi)\left(\frac{u_i'(c_i(\xi,\tau))}{\int \Pr(d\xi)\, u_i'(c_i(\xi,\tau))} \times \frac{dc_i(\xi,\tau)}{d\tau}\right)\right\}}_{\text{DS-redistribution}}. \qquad (26)
\end{aligned}$$

Most of our criticism of the BF decomposition also applies to Davila and Schaab's decomposition. Both decompositions do not satisfy Properties (a), (b), and (c) and, in the attempt to have Pareto weights appear only in the redistribution component, do not aggregate gains from aggregate efficiency, insurance, and redistribution consistently.[11] However, the Davila and Schaab decomposition differs from ours and BF decompositions even in settings in which the latter two coincide. Thus, it would interpret the sources of welfare gains from a marginal

[10]The problem highlighted above is not limited to aggregating of gains from insurance. In the appendix, we construct a related example to show that the efficiency component of a welfare decomposition must depend on social weights as well.

[11]Davila and Schaab refer to the "insurance" component as the "risk-sharing" and their applications primarily focus on risk-sharing of aggregate shock across heterogeneous households. The modification of our example from Section 4.3 shows that such gains must also be generically Pareto-weighted in welfare decompositions. Suppose there are three groups of households. Group 1 faces no ex-post risk, while ex-post risk to groups 2 and 3 being perfectly negatively correlated. Introducing Arrow securities in such settings unambiguously improves risk-sharing between groups 2 and 3, but its contribution to welfare change will depend on social weights. In particular, if groups 2 and 3 have Pareto weights of zero, welfare is unaffected by introducing Arrow securities, and Davila and Schaab's decomposition would need to assign some effect to redistribution or efficiency to offset the positive contribution of their "risk sharing" component.

change in tax progressivity in Benabou's economy differently than the other two decompositions. This implies that comparing allocations at two dates along an exogenous balanced growth path with CRRA preferences – a standard macroeconomic benchmark – ours and BF decomposition would assign 100% of welfare gains to improvements in aggregate efficiency. In contrast, Davila and Schaab's decomposition will assign these gains to a mixture of the three components.[12] This makes Davila and Schaab's decomposition harder to interpret.[13]

5 Applications

In this section, we study fiscal reforms in two calibrated incomplete markets economies. The first application is an income tax reform in the spirit of Aiyagari (1995). In this application, we investigate sources of welfare gains, approximation errors in our decomposition, and how our results compare to ones those emerge from the BF decomposition. Our second application considers a reform in the spirit of Blanchard (2019) that increases public debt in times of low interest rates.

5.1 Income tax reform

Environment Time is discrete and unending. There is a unit measure of households who are heterogeneous ex-ante and subject to idiosyncratic productivity shocks ex-post. Household i has some initial assets $a_{i,0}$ and a vector that summarizes idiosyncratic shocks $\epsilon_{i,0} = \left(\epsilon^P_{i,0}, \epsilon^T_{i,0}\right)$. Random variables $\epsilon^P_{i,t}$ and $\epsilon^T_{i,t}$ correspond to permanent and transitory components of earnings and obey

$$\log \epsilon^P_{i,t} = \rho^P \log \epsilon^P_{i,t-1} + \sigma^P \eta^P_{i,t},$$

$$\log \epsilon^T_{i,t} = \sigma^T \eta^T_{i,t},$$

[12] This discrepancy is ultimately driven by the stance on what to consider to be the "pure aggregate efficiency" change in allocations. Ours, Benabou's and Floden's notions of pure aggregate efficiency is multiplicative: scaling of consumption of all households by the same amount is interpreted as a 100% change in the aggregate efficiency. In contrast, Davila and Schaab' notion is additive: adding the same amount of consumption to each household is interpreted as a 100% change in the aggregate efficiency. While a priori it unclear whether multiplicative or additive property is more desirable, one should note that many standard macroeconomic models have the multiplicative property. In addition to the balanced growth path benchmark mentioned in the text, all relevant economic elasticities are defined with respect to the multiplicative changes and thus fit naturally into ours, Benabou's and Floden's convention.

[13] Davila and Schaab also emphasize that efficiency and insurance properties of their decomposition are invariant to positive affine transformations of the utility functions. It is unclear why such invariance (and not, for instance, any increasing transformation) is a desirable property. More generally, all decompositions of changes in a social welfare function such as the one specified in equation (1) necessarily require a researcher to take stands on the cardinality of the utility function as well as interpersonal comparisons of utilities.

where $\left\{\eta_{i,t}^P, \eta_{i,t}^T\right\}$ are standard Gaussian shocks independent of each other and $\rho^P, \sigma^P, \sigma^T$ are parameters that capture persistence and volatility. Household i solves

$$\max_{c_{i,t}, n_{i,t}, a_{t+1}} \mathbb{E}_0 \sum_t \beta^t \left[\frac{c_{i,t}^{1-\sigma}}{1-\sigma} - \chi \frac{n_{i,t}^{1+\gamma}}{1+\gamma}\right]$$

subject to

$$c_{i,t} + a_{i,t+1} = (1-\tau)\left[r_t a_{i,t} + w_t \epsilon_{i,t}^P \epsilon_{i,t}^T n_{i,t}\right] + a_{i,t} + T_t,$$

$$a_{i,t+1} \geq \underline{a},$$

given initial conditions $a_{i,0}, \epsilon_{i.0}$. Here, τ is a proportional tax on the sum of capital and labor earnings and T_t is a uniform lump sum transfer.
Let $A_t = \int a_{i,t} di$, $L_t = \int \epsilon_{i,t}^P \epsilon_{i,t}^T n_{i,t} di$, $C_t = \int c_{i,t} di$ be aggregate savings, consumption, and labor. Aggregate savings are allocated between capital and government debt D

$$K_{t+1} + D = A_{t+1}.$$

A Cobb-Douglas technology $Y_t = A K_t^\theta L_t^{1-\theta}$ is owned by a representative firm which hires labor and capital at factor prices $\{r_t, w_t\}$. A feasible allocation at time t satisfies

$$Y_t = C_t + G + K_{t+1} - (1-\delta) K_t,$$

where G is government expenditures and δ is a depreciation rate. Given initial distribution μ_0 and fiscal policy $\{\tau, D, G\}$, a competitive equilibrium is an allocation $\{c_{i,t}, n_{i,t}, a_{i,t+1}\}_{i,t}$, sequence of prices $\{r_t, w_t\}_t$, and transfers $\{T_t\}_t$ such that individuals and firms optimize and markets clear.

We study a permanent change in the tax rate τ. For a given $\{D, G\}$, we pick τ^A and compute the invariant distribution $\mu^{A,inv}$. Our baseline economy A corresponds to $\mu_0 = \mu^{A,inv}$ and $\left\{\tau^A, D, G\right\}$. We study the welfare effects of switching to some policy $\tau^B \neq \tau^A$ that results in an equilibrium associated with $\mu_0 = \mu^{A,inv}$ and $\left\{\tau^B, D, G\right\}$. This experiment is a simple way to capture a popular approach for analyzing tax forms. Under such an approach, policy A corresponds to a current U.S. system and the invariant distribution $\mu^{A,inv}$ is meant to represent the current U.S. economy. Policy B is either some specific alternative or an "optimal" policy under some postulated welfare criterion.[14] We capture

[14] See for instance Huang et al. (1997), Aiyagari and McGrattan (1998), Conesa and Krueger (1999), Floden (2001), Domeij and Heathcote (2004), Meh (2005), Erosa and Koreshkova (2007), Conesa et al. (2009), Krueger and Ludwig (2013), Gottardi et al. (2015), Krueger and Ludwig (2016), Heathcote et al. (2017),

this approach by varying a unidimensional variable τ^B.

We use formula (13) to study the source of the welfare change from this policy reform. In the present setting, consumption and labor at different dates correspond to different goods. Expressions for each component become

$$\begin{aligned}
\text{agg. efficiency} &= \sum_t \int \left[\phi_{c,t}\left(a_0,\epsilon_0\right)\Gamma_{c,t}+\phi_{n,t}\left(a_0,\epsilon_0\right)\Gamma_{n,t}\right] d\mu_0, \\
\text{redistribution} &= \sum_t \int \left[\phi_{c,t}\left(a_0,\epsilon_0\right)\Delta_{c,t}\left(a_0,\epsilon_0\right)+\phi_{n,t}\left(a_0,\epsilon_0\right)\Delta_{n,t}\left(a_0,\epsilon_0\right)\right] d\mu_0, \\
\text{insurance} &= \sum_t \int \left[\phi_{c,t}\left(a_0,\epsilon_0\right)\sigma\Lambda_{c,t}\left(a_0,\epsilon_0\right)-\phi_{n,t}\left(a_0,\epsilon_0\right)\gamma\Lambda_{n,t}\left(a_0,\epsilon_0\right)\right] d\mu_0.
\end{aligned}$$

All terms can be quickly and efficiency obtained using objects that are computed when constructing standard incomplete market equilibria. We provide details in the appendix B.1.[15]

Calibration We follow a standard approach to parameterize our economy. We set $\sigma = 1$ and $\gamma = 2$, and choose the labor disutility parameter χ to obtain average hours equal to 1/3. The subjective discount factor β is set to 0.96 to generate an after-tax return on capital (net of growth) of about 3%. We set capital share and depreciation rate (θ, δ) to $(0.36, 0.1)$ in order to deliver a capital to output ratio of about 2.5 and an investment rate of 10%. We set baseline fiscal policy parameters $\left(\tau^A, B, G\right)$ to target a marginal income tax rate of 30%, public debt to output of 100%, and government spending (excluding transfer payments) to output equal to 15%. We adopt Krueger et al. (2016)'s choices for $\left(\rho^P, \sigma^P, \sigma^T\right)$. This calibration generates a standard deviation of log wage earnings of 55% and a standard deviation of assets that is 1.6 times the average per capita asset holdings. All calibrated parameters are listed in Table 1. The first column of Table 2 reports various moments of the invariant distribution.

For our baseline reform consider a modest increase in income tax from $\tau^A = 0.3$ to $\tau^B = 0.33$. Figure 1 plots transition paths for key aggregates under this tax reform. Higher taxes increase tax revenues and transfers, and lead to a reduction in labor supply and capital.

Rohrs and Winter (2017), McGrattan and Prescott (2017), Acikgöz et al. (2018), Hosseini and Shourideh (2019), Boar and Midrigan (2019), Boar and Midrigan (2020), Chien and Wen (2021), Bruggemann (2021), Dyrda and Pedroni (2021).

[15] A key object that appears in many terms is the future expectation of individual policy variables conditioned on states in date $t = 0$, $\mathbb{E}\left[x_t^j | a_0, \epsilon_0\right]$. In the appendix, we provide an fast procedure to compute such expectations using only pre-solved policy functions and thereby avoiding errors due to Monte Carlo simulations.

TABLE 1: CALIBRATION

Parameters		Values
Preferences		
risk aversion	σ	1
subjective discount factor	β	0.96
labor supply	(γ, χ)	$(2, 17.89)$
Production		
capital share	θ	0.36
depreciation rate	δ	0.1
Shocks		
persistent component	$\left(\rho^P, \sigma^P\right)$	$(0.9695, 0.1959)$
transitory component	σ^T	0.23
Policy		
tax rate	τ	0.30
debt/GDP	D/Y	1
govt. spending/ GDP	G/Y	0.15

Notes: Parameters for the baseline economy.

Labor adjusts much faster than capital. As the capital stock slowly decreases, wages increase. This leads leads to a slight increase in labor supply after a sharp initial drop. Eventually, the economy under tax τ^B converges to a new stationary distribution $\mu^{B,inv}$ that we summarize in the second column of Table 2.

Decomposition The welfare effect of this policy change depends on Pareto weights. Figure 2 plots the welfare, $\mathcal{W}^B - \mathcal{W}^A$ and its components for two one-parameter families of Pareto weights, $\alpha(a_0, \epsilon_0) \propto \exp(\alpha_\epsilon \epsilon_0)$ and $\alpha(a_0, \epsilon_0) \propto \exp(\alpha_a a_0)$ normalized so that $\int \alpha(a_0, \epsilon_0)\, d\mu_0 = 1$. Here α_ϵ and α_a are parameters that expresses relative weights on households with higher earnings and asset income. A Utilitarian planner would set $\alpha_\epsilon = 0$ and $\alpha_a = 0$.[16]

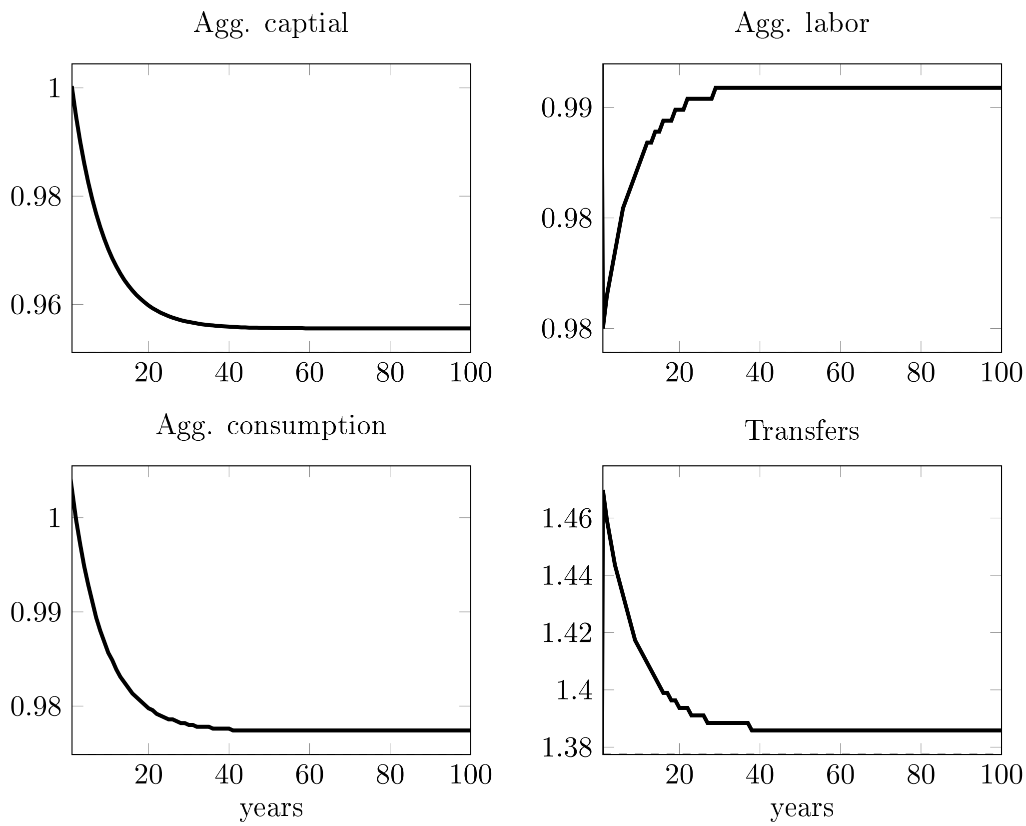
[16] By loosely appealing to a "behind of veil of ignorance" argument, it is sometime argued that the Utilitarian criterion is a natural benchmark. That justification is problematic in Aiyagari-style economies. The initial distribution at the time of the reform, $\mu^{A,inv}$, reflects past shocks that households experienced, so households' asset holdings are proportional to forgone past consumptions. Applying a Utilitarian criterion to evaluate the effect of policy changes at endogenous invariant distributions formed under different policies ignores those past shocks and is inconsistent with a "behind the veil of ignorance" spirit. If a policy reform produces both winners and losers, without discussing how heterogeneities among households arise in the first place, it is generally impossible to justify a unique "correct" welfare criterion. We use a Utilitarian

TABLE 2: STEADY STATES

Moments	Economy A	Economy B
Aggregates		
pvt. consumption/GDP	59.58%	59.71%
investment/GDP	25.42%	24.90%
Factors		
hours	33.00%	32.58%
capital/GDP	2.54	2.49
Govt.		
tax revenues/GDP	23.62%	26.29%
interest payments/GDP	4.16%	4.57%
transfers/GDP	4.46%	6.34%
govt. spending/GDP	15.00%	15.38%
Households		
std./mean of consumption	74.78%	73.08%
std./mean of assets	160.22%	161.07%
std./mean of wage earnings	76.64%	74.06%
corr of wage earnings and assets	43%	42%

Notes: Steady states for economy A and economy B. The parameters for Economy A are described in 1. The parameters in Economy B are same as Economy A except τ^B is set to 0.33

Figure 1: TRANSITIONS AFTER TAX REFORM



Notes: Transition paths after a change in the income-tax rate from $\tau^A = 0.3$ to $\tau^B = 0.33$. All variables are normalized by their steady-state values in Economy A.

The black solid line on Figure 2 shows the welfare effect $\mathcal{W}^B - \mathcal{W}^A$. The sign of this effect depends critically on Pareto weights used to evaluate welfare. Tax τ^B increases welfare if higher weight is placed on earnings- and asset-poor households (low values of α_ϵ and α_a) and decreases welfare otherwise. The intuition for that can be seen clearly from the three components of the welfare decomposition. The redistributive component of welfare (red lines) is highly sensitive to the choice of the welfare criterion and can take large positive or negative values depending on the values of α_ϵ and α_a. The other two components are much less sensitive to Pareto weights. The insurance component (gray lines) is always positive with higher taxes, but its magnitude decreases in α_ϵ and α_a. This is because assets provide additional means of insurance to households against idiosyncratic earnings shocks. Since earnings and assets are positively correlated in our model, a household with high earnings or high assets values insurance from higher taxes less than a household with low earnings and asset income. Finally, the efficiency component is quantitatively small. One might expect that higher taxes reduce aggregate efficiency because they introduce higher distortions. But this need not be so since such taxes also undo inefficiencies associated with households' incentives to oversave, an effect emphasized by Aiyagari (1995).

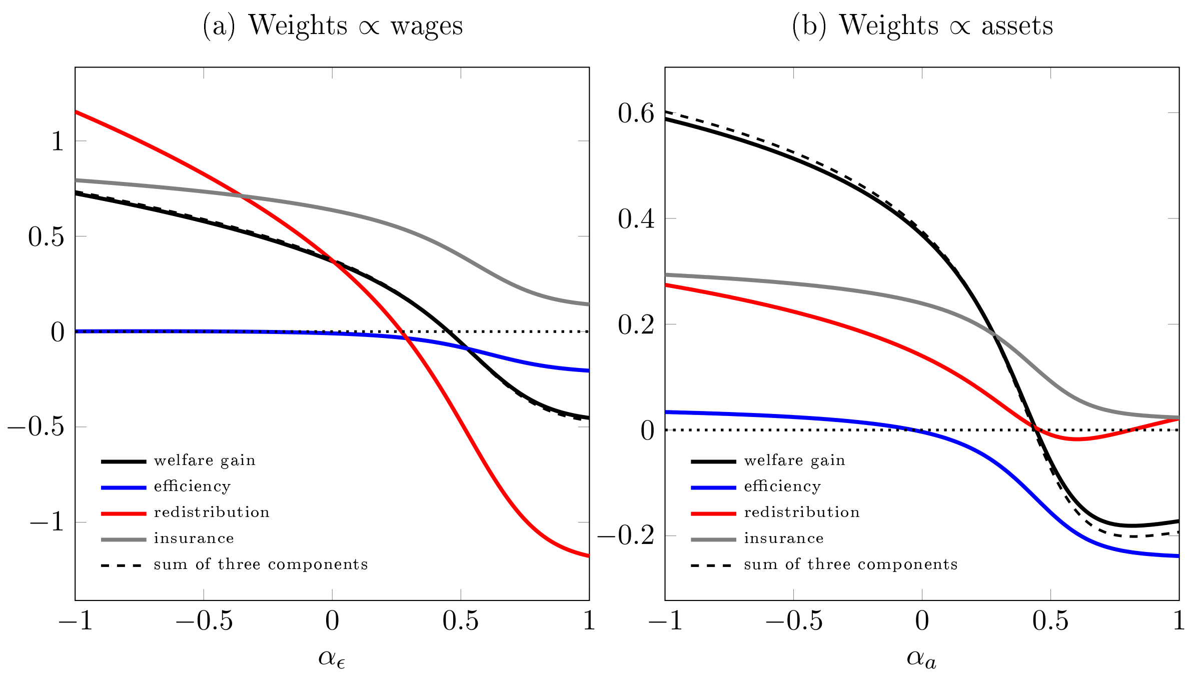
The dotted black line on Figure 2 is the sum of the three components in our decomposition from equation (13). It is virtually indistinguishable from the solid black line, which is the left hand side of (13). Thus, the approximation error in our decomposition is small for all welfare weights that we have considered.

Dependence on size of tax change We now study how our results depends on the size of the tax change. We use the Utilitarian criterion and in Figure 3 plot our decomposition as a function of τ^B.

The insurance and redistribution components are both monotonic in τ^B in the range we consider. The relative magnitude of these two components is is roughly independent of τ^B. The aggregate efficiency component is inversely U-shaped and very sensitive to τ^B. This outcome is driven by two offsetting forces of higher taxes on efficiency: they increase inefficiency by distorting the consumption-leisure decision, and they reduce over-saving. The two effects cancel in our calibration at a value of τ^B close to the chosen value of τ^A. Aggregate efficiency is negative for low values of τ^B because of the over-saving effect and for high values of τ^B because of the consumption-leisure distortion. Approximation errors

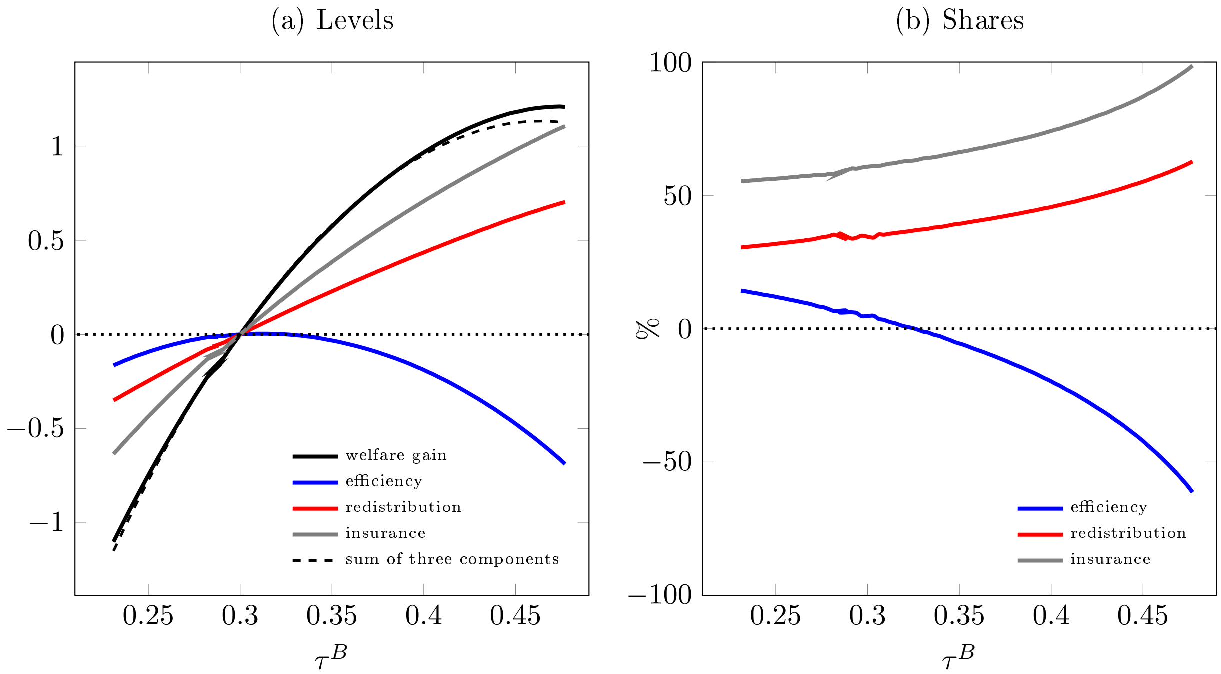
benchmark below merely as an example. We do not assert that it has a better theoretical justification over other values of α_ϵ or α_a.

Figure 2: COMPONENTS OF WELFARE CHANGE ACROSS PARETO WEIGHTS



Notes: Welfare gain and its components for different Pareto weights. In the left panel, Pareto weights $\alpha(a_0, \epsilon_0) \propto \exp\{\alpha_\epsilon e(\epsilon_0)\}$ with $\alpha_\epsilon \in [-1, 1]$ and in the right panel, Pareto weights $\alpha(a_0, \epsilon_0) \propto \exp\{\alpha_a a_0\}$ with $\alpha_a \in [-1, 1]$.

Figure 3: COMPONENTS OF WELFARE CHANGE ACROSS SIZE OF TAX CHANGE



Notes: Welfare decomposition and approximation errors for $\tau^B \in [0.2, 0.5]$. In the left panel, we plot the levels of the three components as well as the true welfare gain. In the right panel, the three lines are the shares of each of the three components of welfare.

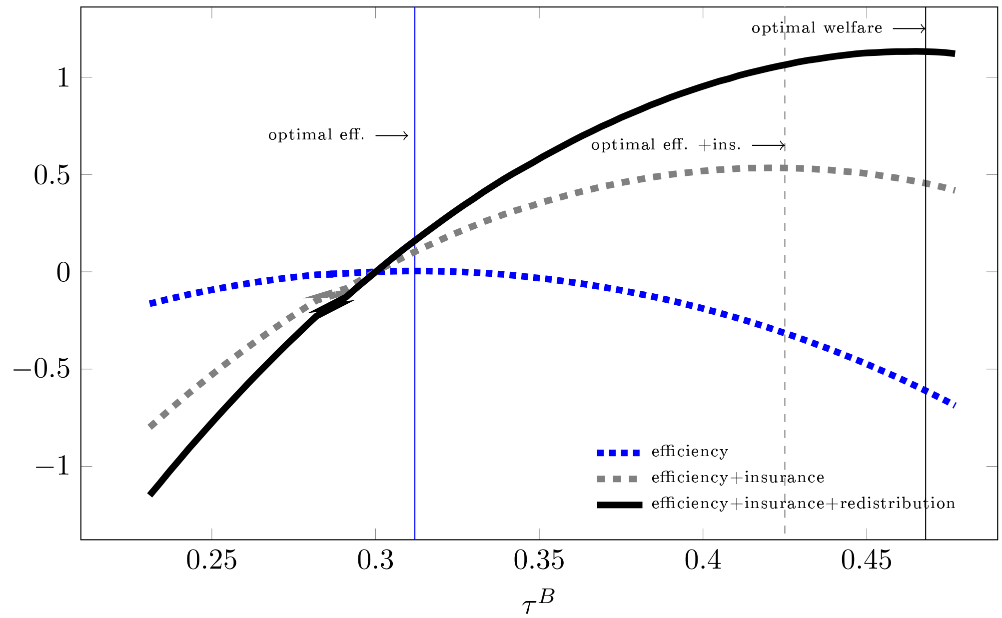
in the decomposition remain low even for very large policy changes. At $\tau^B = 0.5$, it is only 7.5% of the overall welfare effect $\mathcal{W}^B - \mathcal{W}^A$.

Component Planners The additive structure of our decomposition leads naturally to the notion of *component planners* who focus on gains from subsets of sources. Given an allocation A, a component planner, indexed with $\iota \equiv \left(\iota^{AE}, \iota^R, \iota^{INS}\right) \in [0,1]^3$ such that $\iota^E + \iota^R + \iota^{INS} = 1$, orders allocation B with

$$\mathcal{W}^{A,B}(\iota) \equiv \iota^{AE}\mathbb{E}\phi_i\Gamma + \iota^R\mathbb{E}\phi_i\Delta_i + \iota^{INS}\mathbb{E}\phi_i\gamma_i\Lambda_i.$$

We calculate optimal tax rates for component planners for our income tax reform. As discussed above, gains from efficiency are non-monotonic. Thus, a component planner who cares only about aggregate efficiency, $\iota = (1, 0, 0)$ must trade off the oversavings externality against labor supply distortions. On the other hand, gains from insurance and redistribution are monotonically increasing in the size of the tax rate. Consequently, a component planner who cares only about insurance or redistribution will want a corner solution with $\tau = 100\%$,

Figure 4: COMPONENT PLANNERS



Notes: Optimal component planners. The vertical blue (dashed) line is the optimal tax for the aggregate efficiency component, the vertical gray (dotted) line is the optimal tax for sum of aggregate efficiency and insurance components, and the vertical black (solid) line is the optimal tax for total welfare.

which allows that component planner to fully redistribute and insure all variation in labor income.

In Figure 4, we study three planners who care about: (i) only aggregate efficiency, so that $\iota = (1, 0, 0)$, (ii) aggregate efficiency and insurance, so that $\iota = (1, 1, 0)$, and (iii) total welfare, so that $\iota = (1, 1, 1)$. Vertical lines plot the maximum tax rates for each component planner. The optimal tax rate that maximizes the aggregate efficiency criteria (i) is 31% , while tax rates that maximize criteria (ii) and (iii) are higher as they internalize positive gains from insurance and redistribution. The optimal tax rate for the aggregate efficiency and insurance planner (ii) is 42.5% while the tax rate that maximizes planner (iii), who cares about all three sources, is 46.8%.

Comparison of welfare using invariant distributions We now use our Section 3.3 welfare decomposition using invariant distributions associated with policies τ^A and τ^B.

Column (a) in Table 3 reproduces our decomposition from Figure 2 under the Utilitarian criterion. Column (b) provides the decomposition of welfare changes between welfare in the steady states corresponding to policies $\left\{\tau^B, G, D\right\}$ and $\left\{\tau^A, G, D\right\}$. A comparison of

TABLE 3: WELFARE DECOMPOSITION

Sources of Welfare Gains	BEGS	BEGS	FLODEN	
		(invariant distributions)	Method 1	Method 2
	(a)	(b)	(c)	(d)
Efficiency	-0.94%	-109%	-122.94%	-122.94%
Redistribution	37.25%	79.10%	116.8%	148.33%
Insurance	63.68%	130.45%	106.05%	72.60%

Notes: In this table we compare our welfare decomposition with two implementations of the Floden decomposition. The column titled "Method 1" computes the consumption certainty equivalent setting labor supply of each individual to using individual policy functions, while the column "Method 2" sets labor supply of each individual to the average labor supply.

columns (a) and (b) reveals several insights. First, losses in aggregate efficiency are much higher when one ignores the transition path and considers only steady state welfare. Figure 1 shows why: the steady-state capital stock is much lower under policy τ^B, but reaching it takes a long time. Most aggregate efficiency losses that occur far in the future are assigned a low weight in column (a) due to discounting. Second, relative importances of insurance and redistribution components are roughly the same under both decompositions, with insurance gains being twice those from redistribution.

It is instructive to compare our decomposition to the BF decomposition. Since that decomposition does not have a natural extension to economies with heterogeneities in both consumption and work hours, Floden proposed two alternative methods that might be used in such settings. Findings from these two methods are reported in columns (c) and (d). They assign similar importances to aggregate efficiency as does our decomposition, but they misrepresent relative importances of insurance and redistribution. Our Section 4.3 discussion explains why. Wealth-rich households value the reduction in labor income risk less than wealth-poor households. Benabou-Floden decomposition views this as redistribution and understates (overstates) the insurance component of a welfare change if households' asset holdings are positively (negatively) correlated with their earnings. As we saw in Table 3, there is a strong positive correlation between the two in our calibration (as in the data), which indicate that Floden's decomposition typically understates the importance of insurance.

5.2 Public debt reform

Blanchard (2019) investigated the policy of running bigger deficits and issuing more public debt when interest rates are low enough. In this section, we use a version of the Blanchard (2019) economic environment with aggregate shocks and incomplete markets calibrated to the distribution of debt returns in the U.S and study a policy reform in which debt is increased by a constant amount and rolled over forever.[17]

Environment Time is discrete and indexed by $t = 1, 2 \ldots \infty$. A technology transforms K_{t-1} units of capital and L_t units of labor into $Z_t F(K_{t-1}, L_t)$ units of output using a constant returns to scale function F. The economy is populated by overlapping generations of two-period lived households and a government. In each period N households are born with e units of good endowment, and $\overline{L}$ units of market time. The government issues Δ_t debt financed by lump sum taxes on (or transfers to) the young. We use x_t^l to denote time t choice of x by an household born at date l.

Given a sequence of wages $\{w_t\}$, returns $\left\{R_t^s, R_t^k\right\}$ on the safe asset and the risky asset, respectively, and taxes $\{T_t\}$, for all dates $t \geq 0$, households solve

$$\max_{c_t^t, k_t^t, b_t^t, c_{t+1}^t} \quad u\left(c_t^t\right) + \beta \mathbb{E}_t v\left(c_{t+1}^t\right)$$

subject to

$$c_t^t + k_t^t + b_t^t = e + w_t \overline{L} - T_t,$$

$$c_{t+1}^t = R_t^s b_t^t + R_{t+1}^k k_t^t.$$

An old individual at date 0 with bond holdings b_{-1}^{-1} and capital k_{-1}^{-1} consumes $c_0^{-1} = R_0^s b_{-1}^{-1} + R_1^k k_{-1}^{-1}$. For all dates $t \geq 0$, goods producing firms solve

$$\max_{K_{t-1}, L_t} Z_t F_k\left(K_{t-1}, L_t\right) - R_t^k K_{t-1} - w_t L_t.$$

Given $\left(k_{-1}^{-1}, b_{-1}^{-1}, A_0\right)$ and a process for public debt $\{\Delta_t\}$, a competitive equilibrium comprises stochastic processes for $\left\{R_t^s, R_t^k\right\}_t$, taxes $\{T_t\}$, allocations $\left\{c_t^{t-1}, c_t^t, k_{t-1}^{t-1}, b_{t-1}^{t-1}\right\}_{t\geq 0}$ such that

[17] A large literature about public debt in low interest rate economies includes Samuelson (1958) and Diamond (1965) in the context of overlapping generation models, Woodford (1990) and Aiyagari (1994) in the context of liquidity properties of public debt, and several papers after Blanchard's AEA presidential address, for instance, Reis (2021), Brumm et al. (2021), Hellwig (2021), Aguiar et al. (2021), Brunnermeier et al. (2022), Barro (2022), and Amol and Luttmer (2022). Most related to our example are Brumm et al. (2021), Hellwig (2021), and Barro (2022), who like us focus on aggregate shocks.

households and firms optimize, with the following market clearing conditions holding for dates $t \geq 0$

$$K_{t-1} = k_{t-1}^{t-1}, \quad \Delta_t = b_{t-1}^{t-1}, \quad L_t = \overline{L},$$

$$c_t^{t-1} + c_t^t + K_t = e + Z_t F\left(K_{t-1}, L_t\right).$$

Reform Our main exercise applies our Section 3 decomposition to a simple version of a proposal studied by Blanchard (2019). For that reason, our economy A will be a calibrated version of our model without a government and in which debt (Δ_t) is set to zero. We then study a reform in which the government issues $\Delta_t = \Delta > 0$ and rolls it forever financing net interest payments by levying a lump sum tax. For initial states (Z_0, K_{-1}) and Pareto weights $\alpha(t)$, welfare is

$$\mathcal{W}(K, Z; \Delta) = \lim_{T \to \infty} \sum_{t=0}^{T-1} \frac{\alpha(t)}{T} \mathbb{E}\left[u\left(c_{t-1}^{t-1}\right) + \beta v\left(c_t^{t-1}\right)\right] | K_{-1} = K, Z_0 = Z, \Delta_t = \Delta$$

with an convention that $u\left(c_{-1}^{-1}\right) = 0$ to specify that the planner ignores consumption before date $t = 0$. This specification maps to Section 3.2 notation when we index goods by age (old and young) and index households by generations.

Calibration We use constant absolute risk aversion preferences $-\exp\{-\lambda c_y\} - \beta \exp\{-\lambda c_o\}$ and assume that the aggregate productivity shocks are Gaussian with mean and standard deviation (μ_Z, σ_Z). We set the non-labor endowment $e = 0$ and normalize the mean of productivity $\mu_Z = 1$. This leaves three parameters, $(\lambda, \beta, \sigma_Z)$. We set λ to target a coefficient of relative risk aversion of 2. We set the parameters (β, σ_Z) to match the standard deviation of return on government debt as well as the fraction of time the cost of borrowing exceeded a steady state rate of growth of output. In the appendix, we report results for alternative values of λ after calibrating to the same set of moments.

To implement this calibration, we use returns on the U.S. debt as the counterpart of $\{R_t^s\}$. The U.S. government issues debt of different maturities, so we target a weighted average measure of returns on the government portfolio. We use bond return and quantity data from CRSP. For the period 1952-2021, we obtain (quarterly) nominal return data on various government bonds of maturities. These data are available for maturities 1 year, 2 year, 5 year, 7 year, 10 year, 20 years, 30 years. We then use the data on bond issuance from CRSP to measure the market value of outstanding bonds by maturity to fit a histogram with bins defined by midpoints located at maturities for which the return data are available. Our measure of returns on debt is $\sum_{bin} \omega_{bin} R_{t\to t+1}^{bin}$ and to construct a measure of returns in excess of growth rate, we deduct the average growth of nominal GDP over the same sample. In Figure 5 we plot the distribution for the weighted-average returns in excess of growth. Average bond returns minus the growth rate over the same sample is close to zero, with an annualized standard deviation of 1.8% quarterly. Furthermore, for about 41% of the quarters, bond returns exceed the mean growth rate.[18]

To map these moments to our model, we treat each period as 20 years, and set $(\beta, \sigma_Z) = (0.95, 0.17)$ to match those moments. We obtain a standard deviation (for 80 quarters length) of R_t^s equal to 16% and the fraction of simulated periods $R_t^s > 1$ equals 40%, all of which are in line with our estimates.[19]

We parameterize $\alpha(t) \propto \exp\{-\alpha_0 \times t/T\}$. We initialize the economy (K_{-1}, A) at the ergodic mean of the baseline (economy A), and then study a small one-time reform that increases $\Delta = 1/2\% \times Y$ for all future periods.

Results We start with $\alpha_0 = 0$, which implements a utilitarian welfare criterion. In column "Utilitarian" of Table 4 we see that the reform *reduces* welfare with negative contributions coming from the efficiency and redistribution components and a positive contribution coming from the insurance component. We calculate component shares by taking ratios of welfare gains attributable to each component divided by the absolute value of the sum.[20] For the baseline calibration, we find a small contribution from redistribution (-0.1%) and offsetting contributions from efficiency (-249.4%) and insurance (+149.5%).

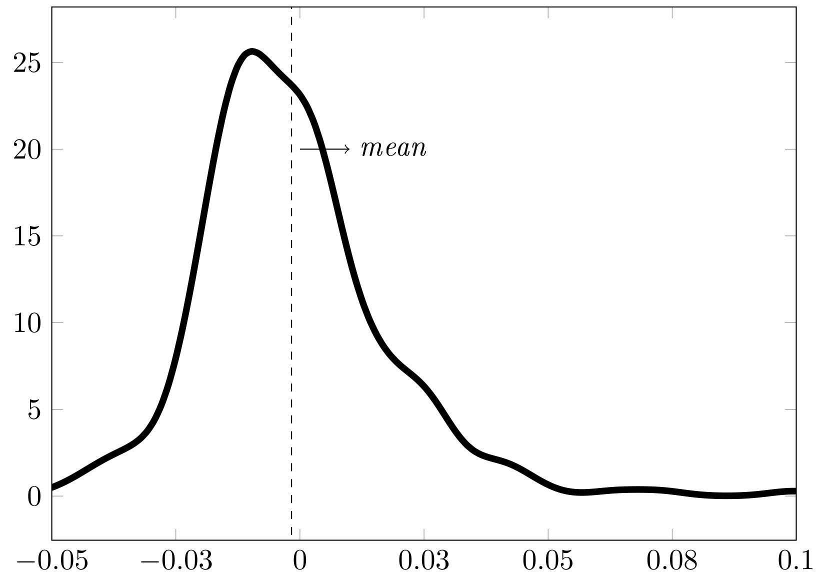
To understand the sources of these findings, we study economies that differ in their average returns on government debt, which we generate by varying the impatience parameter $\beta \in [0.8, 1.20]$. In the left panel of Figure 6, we plot welfare gains (left axis) and average returns on capital R^k (right axis) against the ergodic average returns R^s (in the pre-reform economy) for different values of β. The dashed vertical line corresponds to the baseline value of β. The average return on capital $\mathbb{E}R^k$ (pink line) is greater than the $\mathbb{E}R^s$. The total welfare gains from the reform (black line) and the gains attributed to the efficiency

[18]Our estimates are in line with Mehrotra and Sergeyev (2021) who also compute returns in excess of growth rates for several advanced economies. They exploit the government budget constraint to impute an annual return on overall public debt. For the U.S., their estimates for the faction of time returns in excess of growth rates are positive vary between 30% and 48% depending on the sample.

[19]The implied mean of the return on debt equals $-.1\%$ which is within two standard errors of its empirical counterpart.

[20]We divide by the absolute value of the sum and not the sum to distinguish overall welfare loss from welfare gains.

Figure 5: DISTRIBUTION OF EXCESS DEBT RETURNS



Notes: The figure plots histogram of returns on debt in excess of the growth rate in the U.S. for the sample period 1952-2021. The returns quarterly are computed by taking a weighted average of nominal holding period returns across different maturities. The growth rate is computed by taking the average quarterly growth in log GDP over the same period.

component (blue line) are positive for low values of $\mathbb{E}R^s$ and negative for high values of $\mathbb{E}R^s$. The insurance component (gray line) is always positive and relatively flat across different values of β. The redistribution component (red line) is small and largely unaffected as we vary β.

The efficiency component captures the effects of the reform on the aggregate economy, stripping out the role of incomplete markets. Without insurance considerations, public debt and capital compete for household savings. When the return on capital R^k is lower than growth rate, issuing debt raises output following standard arguments about dynamic efficiency by Diamond (1965). In our setting, the average return on capital exceeds the return on bonds in order to compensate households for bearing risk. Our baseline calibration yields an average return on capital that is greater than one while the return on debt is smaller than one. In Figure 4, we see that if β becomes large enough that the average return on bonds and capital are both lower than one, then the efficiency component (blue line) turns positive.

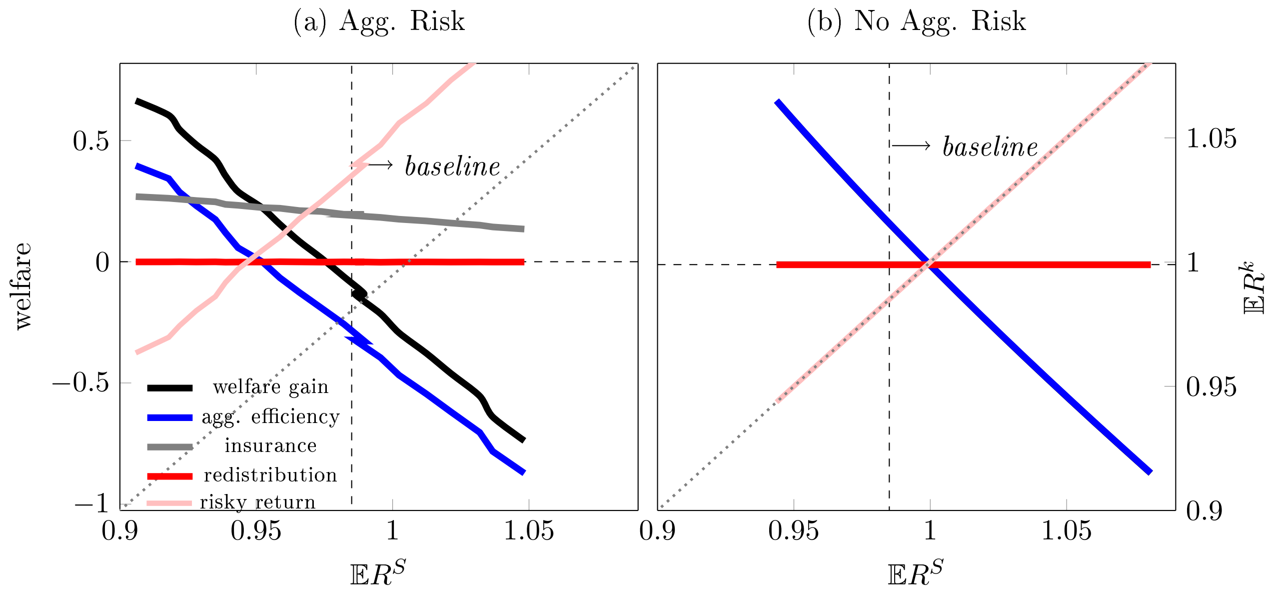
The insurance component captures the force that issuing risk-free debt allows younger households to hedge aggregate risk better when they turn old. This force implies a gain in welfare coming from better insurance across households. For the calibrated economy, we find the contribution from insurance is positive. The gray line labeled "insurance" in the left panel of Figure 6, shows that for the range of β considered in Figure 6, the insurance component remains positive.

Next, we turn to the redistribution component. The red lines labeled "redistribution" in both panels of Figure 6 indicate that the contribution of redistribution is small. The source of welfare gains/losses from redistribution is the across-generation transfers of resources. In the long run, the reform raises the return on capital while it lowers total capital and output. This makes later generations worse off relative to initial generations. The size of the welfare change induced by this intergenerational transfer depends on two things—(i) the speed of transition to the ergodic distribution after the reform, and (ii) how the planner weighs early generations relative to later ones. For our calibrated economy, in which shocks are independent across time and capital depreciates 100%, the speed of transition is quite fast. This makes the contribution of redistribution governed by the shape of $\alpha(t)$ small in the utilitarian case. In columns "More weight on initial generations" and "Less weight on initial generations" of Table 4, we describe the decomposition for two values of parameter $\alpha_0 \neq 0$. Negative (positive) values reflect more (less) weight on the initial generation relative to the utilitarian benchmark. Evidently, the redistribution component is increasing in the Pareto weight attached to the initial old.

Finally, we discuss the role of aggregate risk by turning off stochastic productivity risk (or $\sigma_Z \to 0$). We repeat the same exercise, that is vary $\beta \in [0.8, 1.2]$, and plot components of welfare in the right panel of Figure 6. We see that welfare gains (black line) switch near $R^s = 1$. By construction, the insurance component is zero and nearly all the welfare gains come from aggregate efficiency. In the absence of risk, the return on capital equals the return on debt. Thus, welfare considerations are solely driven by whether any of the returns are below one. As mentioned before, the presence of aggregate risk generates a wedge between the return on capital and return on bonds. In the realistic case in which the return on capital is greater than the growth rate, welfare gains from issuing debt reflect a tradeoff between crowding out capital and better insurance.[21] We find that while the insurance force is positive, it is not enough to offset the crowding out of capital unless the return to capital is sufficiently low. In the appendix, we show that these conclusions are stronger when we recalibrate the model to have larger risk-premia.

[21] These conclusions are in line with numerical simulations in Blanchard (2019) and in Brumm et al. (2021). Our goal here is to interpret these outcomes in light of our proposed decomposition.

Figure 6: WELFARE DECOMPOSITIONS ACROSS DISCOUNT RATES



Notes: Ergodic average of debt returns and and components of welfare across values the discount factor $\beta \in [0.8, 1.2]$. In the right panel (deterministic) we set $\sigma_Z = 0$ and in the left panel (stochastic) we set σ_Z to its value in the baseline calibration.

TABLE 4: WELFARE DECOMPOSITIONS ACROSS PARETO WEIGHTS

	More weight on initial generations	Utilitarian	Less weight on initial generations
	$\alpha_0 = -1$	$\alpha_0 = 0$	$\alpha_0 = 1$
Agg. efficiency	-427.0%	-249.4%	-193.9%
Insurance	266.5%	149.5%	113.7%
Redistribution	60.5%	-0.1%	-19.8%

Notes: Component shares of welfare gains for different values of Pareto Weights (α_0). For a given of α_0, we divide the welfare gains from the baseline reform attributed to each of the component by absolute value of the sum of all three components.

6 Conclusion

We developed a decomposition of welfare changes into three components: aggregate efficiency, which captures effects from changes in the aggregate quantity of resources; redistribution, which captures effects from changes in shares of resources that *ex-ante* heterogeneous households can expect to receive; and insurance, which captures effects of changes in the uncertainties that households face. Our decomposition applies to a large class of multi-person, multi-good, multi-period economies with general specifications of preferences and shocks and sources of heterogeneity.

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A Appendix

A.1 Taylor series in abstract spaces

Recall some properties of Taylor series in general spaces. Let $f : X \to \mathbb{R}$ be a mapping from a normed space X (with some norm $||\cdot||$) into $\mathbb{R}$. The first-order Frechet derivative at a point $x \in X$ is a linear mapping $f'(x) : X \to \mathbb{R}$ such that for each $h \in X$, we have $\lim_{||h||\to 0} \frac{|f(x+h)-f(x)-f'(x)\cdot h|}{||h||} = 0$. The second-order Frechet derivative is the Frechet derivative of $f'(x)$. It is a bilinear map. Any function f that is twice Frechet differentiable satisfies

$$f(x+h) = f(x) + f'(x)\cdot h + \frac{1}{2}f''(x)\cdot(h)^2 + R^f(x,h), \tag{27}$$

where $R^f(x,h)$ is a residual that is of the order $o\left(||h||^2\right)$ (see Cartan (1971), Theorem 5.6.3). When $X \subset \mathbb{R}^n$, functionals $f'(x)$ and $f''(x)$ are simply the Jacobian and Hessian of f, respectively.

For any given $(x,h) \in X \times X$, define function $g : \mathbb{R} \to \mathbb{R}$ by $g(\sigma) \equiv f(x+\sigma h)$. Take Taylor expansion of g around $\sigma = 0$ to get

$$g(1) = g(0) + g'(0) + \frac{1}{2}g''(0) + R^g(1). \tag{28}$$

Since we have $g'(0) = f'(x)\cdot h$ and $g''(0) = f''(x)\cdot(h)^2$, we have

$$R^g(1) = R^f(x,h) = o\left(||h||^2\right). \tag{29}$$

Finally, we use "$\simeq$" to denote that two relationships are equal up to any term of order $o\left(||h||^2\right)$. In this notation, relationship (27) and (28) can be rewritten as

$$\begin{aligned} f(x+h) &\simeq f(x) + f'(x)\cdot h + \frac{1}{2}f''(x)\cdot(h)^2, \\ g(1) &\simeq g(0) + g'(0) + \frac{1}{2}g''(0). \end{aligned}$$

A.2 Conventions and terminology

We define some conventions to be used throughout our proofs. For expressions that take the same value for policies $j = A$ and $j = B$ we occasionally use a shorthand "t.i.p.", meaning "terms independent of policy". By the Law of Iterated Expectations (LIE), we mean the property that for any deterministic function x_i of i and random variable ε_i with $\mathbb{E}_i\varepsilon_i = 0$, we have $\mathbb{E}x_i\varepsilon_i = \mathbb{E}\mathbb{E}_i x_i\varepsilon_i = \mathbb{E}x_i\mathbb{E}_i\varepsilon_i = 0$, and that, by analogous arguments,

$\mathbb{E}x_i\left(\varepsilon_i\right)^2 = \mathbb{E}x_i var_i\left(\varepsilon_i\right)$. We use shorthands $U_i, U_{c,i}, U_{cc,i}$ for $U_i\left(c_i^Z\right), U_{c,i}\left(c_i^Z\right), U_{cc,i}\left(c_i^Z\right)$ in a one-good economy, and $U_i, U_{k,i}, U_{km,i}$ for $U_i\left(\left\{c_{k,i}^Z\right\}_k\right), U_{k,i}\left(\left\{c_{k,i}^Z\right\}_k\right), U_{km,i}\left(\left\{c_{k,i}^Z\right\}_k\right)$ in multi-good economies.

A.3 Derivations of equations (5), (8) and (9)

Equation (5) Let $\Gamma^j \equiv \ln C^j - \ln C^Z$ and $\Delta_i^j \equiv \ln w_i^j - \ln w_i^Z$. Observe that by construction we have

$$\Gamma^B = -\Gamma^A = \frac{1}{2}\Gamma,\ \ \Delta_i^B = -\Delta_i^A = \frac{1}{2}\Delta_i, \tag{30}$$

and therefore

$$\left(\Gamma^j\right)^2, \left(\Delta_i^j\right)^2, \Gamma^j\Delta_i^j \text{ are t.i.p. for all } i.$$

Let $\mathcal{W}^Z \equiv \mathbb{E}\alpha_i U_i\left(c_i^Z\right)$ and write

$$\mathcal{W}^B - \mathcal{W}^A = \left(\mathcal{W}^B - \mathcal{W}^Z\right) - \left(\mathcal{W}^A - \mathcal{W}^Z\right). \tag{31}$$

We now apply Taylor series from Section A.1 to $\left(\mathcal{W}^B - \mathcal{W}^Z\right)$ and $\left(\mathcal{W}^A - \mathcal{W}^Z\right)$. We can write

$$\mathcal{W}^j - \mathcal{W}^Z = \mathbb{E}\alpha_i U\left(\exp\left(\Gamma^j + \Delta_i^j\right)\left(1+\varepsilon_i^j\right)c_i^Z\right) - \mathbb{E}\alpha_i U\left(c_i^Z\right).$$

In the language of section A.1, the space X consists of sequences and stochastic processes $\left\{\tilde{\Gamma}, \tilde{\Delta}_i, \tilde{\varepsilon}_i\right\}_i$. We have that $\tilde{\Gamma} \in \mathbb{R}$ and $\left\{\tilde{\Delta}_i\right\}_i$ is a mapping from $[0,1]$ to $\mathbb{R}$. We can represent stochastic processes $\tilde{\varepsilon}_i$ as mappings $\tilde{\varepsilon}_i(\xi)$ with distribution $\Pr_i(d\xi)$ where without loss of generality $\xi \in [0,1]$. Thus, $\{\tilde{\varepsilon}_i\}_i$ maps from $[0,1]^2$ to $\mathbb{R}$. Any $x \in X$ can be represented as $x = \left(\tilde{\Gamma}, \tilde{\Delta}, \tilde{\varepsilon}\right) \in \mathbb{R} \times L^2([0,1]) \times L^2\left([0,1]^2\right)$. We endow X with a corresponding norm. We define function $f : X \to R$ by $\mathbb{E}\alpha_i U\left(\exp\left(\tilde{\Gamma}^j + \tilde{\Delta}_i^j\right)\left(1+\tilde{\varepsilon}_i^j\right)c_i^Z\right) - \mathbb{E}\alpha_i U\left(c_i^Z\right)$. The analogue to $g(\sigma)$ is

$$\mathcal{W}^j(\sigma) = \mathbb{E}\alpha_i U\left(\exp\left(\sigma\left(\Gamma^j + \Delta_i^j\right)\right)\left(1+\sigma\varepsilon_i^j\right)c_i^Z\right).$$

To apply (27) and (28), we set $x = (0,0,\mathbf{0})$ and $h = \left(\Gamma^j, \Delta^j, \varepsilon^j\right)$. Applying (28) and

(29) we get

$$\begin{aligned}\mathcal{W}^j - \mathcal{W}^Z &= \mathbb{E}\alpha_i U_{c,i} c_i^Z \left(\Gamma^j + \Delta_i^j\right) + \frac{1}{2}\mathbb{E}\alpha_i U_{c,i} c_i^Z \left(\Gamma^j + \Delta_i^j\right)^2 \\ &\quad + \frac{1}{2}\mathbb{E}\alpha_i U_{cc,i} \times \left(c_i^Z\right)^2 \left[\left(\Gamma^j + \Delta_i^j\right)^2 + \left(\varepsilon_i^j\right)^2\right] + o\left(\left\|\Gamma^j, \Delta^j, \varepsilon^j\right\|^2\right) \\ &\simeq \mathbb{E}\alpha_i U_{c,i} c_i^Z \left(\Gamma^j + \Delta_i^j\right) + \frac{1}{2}\mathbb{E}\alpha_i U_{cc,i} \times \left(c_i^Z\right)^2 var_i \left(\varepsilon_i^j\right) + \text{t.i.p.} \qquad (32)\end{aligned}$$

Here the expression in the last line is obtained by applying the LIE and dropping $o\left(\left\|\Gamma^j, \Delta^j, \varepsilon^j\right\|^2\right)$ terms from the first expression on the right side of (32).

Substitute (32) into (31) to get

$$\mathcal{W}^B - \mathcal{W}^A \simeq \mathbb{E}\underbrace{\alpha_i U_{c,i} c_i^Z}_{\equiv \phi_i} \Gamma + \mathbb{E}\underbrace{\alpha_i U_{c,i} c_i^Z}_{\equiv \phi_i} \Delta_i + \frac{1}{2}\mathbb{E}\underbrace{\alpha_i U_{c,i} c_i^Z}_{\equiv \phi_i} \underbrace{\frac{U_{cc,i} c_i^Z}{U_{c,i}}}_{\equiv -\gamma_i} \left[\mathbb{E}_i \left(\varepsilon_i^B\right)^2 - \mathbb{E}_i \left(\varepsilon_i^A\right)^2\right]. \quad (33)$$

Finally, observe that

$$var_i \left(\ln c_i^j\right) = \mathbb{E}_i \left(\ln\left(1 + \varepsilon_i^j\right) - \mathbb{E}_i \ln\left(1 + \varepsilon_i^j\right)\right)^2 \simeq \mathbb{E}_i \left(\varepsilon_i^j\right)^2. \quad (34)$$

Substitute this relationship and the definitions of γ_i, ϕ_i into (33) to obtain (5).

Equation (8) Written in σ notation we have

$$U(c_i^{ce,j}(\sigma)) = \mathbb{E}U\left(\exp\left(\sigma\left(\Gamma^j + \Delta_i^j\right)\right)\left(1 + \sigma\varepsilon_i^j\right) c_i^Z\right).$$

A second-order expansion of both sides yields

$$\begin{aligned}& U(\bar{c}_i^{ce,j}) + U'(\bar{c}_i^{ce,j})\bar{c}_{i,\sigma}^{ce,j} \\ & + 0.5\left(U'(\bar{c}_i^{ce,j})\bar{c}_{i,\sigma\sigma}^{ce,j} + U''(\bar{c}_i^{ce,j})\left(\bar{c}_{i,\sigma}^{ce,j}\right)^2\right) \\ & \simeq U(c_i^Z) + U'(c_i^Z)c_i^Z\left(\Gamma^j + \Delta_i^j\right) \\ & \quad + 0.5\left(U''\left(c_i^Z\right)\left(c_i^Z\left(\Gamma^j + \Delta_i^j\right)\right)^2 + U''(c_i^Z)\left(c_i^Z\right)^2 \mathbb{E}_i\left(\varepsilon_i^j\right)^2 + U'(c_i^Z)c_i^Z\left(\Gamma^j + \Delta_i^j\right)^2\right).\end{aligned}$$

Combining like terms yields

$$\bar{c}_i^{ce,j} = c_i^Z$$
$$\bar{c}_{i,\sigma}^{ce,j} = c_i^Z \left(\Gamma^j + \Delta_i^j\right)$$
$$\bar{c}_{i,\sigma\sigma}^{ce,j} = c_i^Z \left(\Gamma^j + \Delta_i^j\right)^2 + \frac{U''(c_i^Z)\left(c_i^Z\right)^2}{U'(c_i^Z)}\mathbb{E}_i\left(\varepsilon_i^j\right)^2 = c_i^Z \left(\Gamma^j + \Delta_i^j\right)^2 - \gamma_i c_i^Z \mathbb{E}_i\left(\varepsilon_i^j\right)^2$$

so

$$c_i^{ce,j} \simeq c_i^Z + c_i^Z \left(\Gamma^j + \Delta_i^j\right) + \frac{1}{2}\left(c_i^Z \left(\Gamma^j + \Delta_i^j\right)^2 - c_i^Z \gamma_i \mathbb{E}_i\left(\varepsilon_i^j\right)^2\right).$$

Similarly, we have that

$$\mathbb{E}_i c_i^j \simeq c_i^Z + c_i^Z \left(\Gamma^j + \Delta_i^j\right) + \frac{1}{2}\left(c_i^Z \left(\Gamma^j + \Delta_i^j\right)^2\right)$$

and thus

$$c_i^{ce,j} \simeq \mathbb{E}_i c_i^j - \frac{1}{2} c_i^Z \gamma_i \mathbb{E}_i \left(\varepsilon_i^j\right)^2 \simeq \mathbb{E}_i c_i^j - \frac{1}{2}\left(\mathbb{E}_i c_i^j\right)\gamma_i \mathbb{E}_i\left(\varepsilon_i^j\right)^2.$$

All together this implies that

$$\ln\left(\frac{c_i^{ce,j}}{\mathbb{E}_i c_i^j}\right) \simeq -\frac{1}{2}\gamma_i \mathbb{E}_i\left(\varepsilon_i^j\right)^2 \simeq -\frac{1}{2}\gamma_i var_i\left(\ln c_i^j\right)$$

Equation (9) We now verify equation (9). As a preliminary step, observe that

$$\frac{w_i^j}{w_i^Z} = \exp\left(\Delta_i^j\right) \simeq \Delta_i^j + \text{t.i.p.} = \ln\frac{w_i^j}{w_i^Z} + \text{t.i.p.} \tag{35}$$

Therefore,

$$\begin{aligned}
\text{redistribution} &= \sqrt{C^A C^B}\mathbb{E}\alpha_i U_{c,i} \ln\frac{w_i^B}{w_i^A}\sqrt{w_i^A w_i^B} = \sqrt{C^A C^B}\mathbb{E}\alpha_i U_{c,i}\left(\ln\frac{w_i^B}{w_i^Z} - \ln\frac{w_i^A}{w_i^Z}\right) w_i^Z \\
&\simeq \sqrt{C^A C^B}\mathbb{E}\alpha_i U_{c,i}\left(\frac{w_i^B}{w_i^Z} - \frac{w_i^A}{w_i^Z}\right) w_i^Z = \sqrt{C^A C^B}\mathbb{E}\alpha_i U_{c,i}\left(w_i^B - w_i^A\right),
\end{aligned}$$

which verifies equation (9).

A.4 Uniqueness

We are interested in additive decompositions of $\mathcal{W}^A - \mathcal{W}^B$ that satisfy Properties (a)–(d) in the main text. We first define the space of additive decompositions. Let X be the space of allocations defined in Appendix A.3. An additive decomposition of $\mathcal{W}^A - \mathcal{W}^{\mathcal{B}}$ consist of three mappings $\mathcal{W}^k : X^2 \to \mathbb{R}$ for $k \in \{E, I, R\}$ such that

$$\mathcal{W}^A - \mathcal{W}^B = \mathcal{W}^E_{A\to B} + \mathcal{W}^I_{A\to B} + \mathcal{W}^R_{A\to B}.$$

Let $L^2\left([0,1], \mathbb{R}^k\right)$ be the space of square-integrable functions $f : [0,1] \to \mathbb{R}^k$. Let $\tilde{X_F} = \mathbb{R} \times L^2([0,1], \mathbb{R}) \times L^2([0,1], \mathbb{R})$ and $\tilde{X_O} = L^2\left([0,1], \mathbb{R}^N\right)$.

We focus on additive decompositions in which each of the mappings $\left\{\mathcal{W}^k_{A\to B}\right\}_{k\in\{E,I,R\}}$ are represented by functions: $F^{i,k} : \tilde{X_F} \times \tilde{X_F} \to \mathbb{R}^N$ and $O^k : \tilde{X_O} \to \mathbb{R}$ such that

$$\mathcal{W}^k_{A\to B}\left(\left\{c^A_i\right\}, \left\{c^B_i\right\}\right) = O^k\left(\left\{\mathbb{E}_i F^{i,k}\left(\left\{C^A, w^A_i, \epsilon^A_i\right\}_i, \left\{C^B, w^B_i, \epsilon^B_i\right\}_i\right)\right\}_i\right). \tag{36}$$

The "inner" functions $F^{i,k}$ map an allocation for some realized states to $\mathbb{R}^N$. We then take expectations of those N outcomes across states with household-specific probability measures. The outer function O^k maps the collection of these expectations into a real number. In this decomposition, we assume that households have identical beliefs over risk so $\mathbb{E}_i\epsilon^j_{\tilde{i}} = 0$ by construction. The class of additive decompositions that satisfy (36) is quite general and includes the decomposition that we propose in Section (3), the decompositions proposed by Benabou (2002), Floden (2001) and the one proposed in Davila and Schaab (2022).

We expand $\mathcal{W}^k_{A\to B}$ around the mid-point allocation c^Z_i. As before, $\Gamma^j \equiv \ln C^j - \ln C^Z$ and $\Delta^j_i \equiv \ln w^j_i - \ln w^Z_i$,and we parameterize allocations using σ

$$\begin{aligned}
&\mathcal{W}^k_{A\to B}(\sigma) \\
&= O^k\left(\left\{\mathbb{E}_i F^{i,k}\left(\left\{\exp\left(\sigma\Gamma^A\right)C^Z, \exp\left(\sigma\Delta^A_i\right)w^Z_i, \sigma\epsilon^A_i\right\}_i, \left\{\exp\left(\sigma\Gamma^B\right)C^Z, \exp\left(\sigma\Delta^B_i\right)w^Z_i, \sigma\epsilon^B_i\right\}_i\right)\right\}_i\right),
\end{aligned}$$

and take Taylor expansion of $\mathcal{W}^k_{A\to B}(\sigma)$ with respect to σ around $\sigma = 0$. Exploiting the fact that the shocks are mean zero, $\mathbb{E}_i\epsilon^j_{\tilde{i}} = 0$, and the c^Z_i is a midpoint, $\frac{1}{2}\Gamma = \Gamma^B = -\Gamma^A$ and $\frac{1}{2}\Delta_i = \Delta^B_i = -\Delta^A_i$, we have that this expansion can be represented by the following quadratic form

$$\begin{aligned}
\mathcal{W}^k_{A\to B} \simeq \mathcal{W}^k_{A\to B,\Gamma}\Gamma + \int \mathcal{W}^k_{A\to B,\Delta}\Delta_i di + \frac{1}{2}\Bigg(&\mathcal{W}^k_{A\to B,\Gamma\Gamma}\Gamma^2 + 2\int \mathcal{W}^k_{A\to B,\Gamma\Delta_i}\Gamma\Delta_i di \\
&+ \iint \mathcal{W}^k_{A\to B,\Delta_i\Delta_{\tilde{i}}}\Delta_i\Delta_{\tilde{i}} d\tilde{i} di + \sum_{j,\tilde{j}}\iint \mathcal{W}^k_{A\to B,\epsilon^j_i\epsilon^{\tilde{j}}_i}\mathbb{E}_i\epsilon^j_i\epsilon^{\tilde{j}}_i d\tilde{i} di\Bigg).
\end{aligned} \tag{37}$$

This must hold for arbitrary Γ,Δ_i and ϵ^j_i. First consider the case where $\Delta_i = 0$ and $\epsilon^j_i = 0$. In order to satisfy property (a) the decomposition must attribute all of the welfare gains to efficiency so from equation (37)

$$\mathcal{W}^E_{A\to B,\Gamma}\Gamma+\frac{1}{2}\mathcal{W}^E_{A\to B,\Gamma\Gamma}\Gamma^2\simeq\mathcal{W}^E_{A\to B}\simeq\mathcal{W}_{A\to B}\simeq\mathbb{E}\phi_i\Gamma.$$

Similarly, when $\Gamma=0$ and $\epsilon_i^j=0$ property (b) implies that all of the welfare gain must be attributed to redistribution so $\mathcal{W}^E_{A\to B}=0$ and hence

$$\int\mathcal{W}^E_{A\to B,\Delta}\Delta_i di+\frac{1}{2}\iint\mathcal{W}^E_{A\to B,\Delta_i\Delta_{\tilde{i}}}\Delta_i\Delta_{\tilde{i}}d\tilde{i}di\simeq 0.$$

Finally, when $\Gamma=0$ and $\Delta_i=0$ all of the welfare gains must be attributed to insurance, $\mathcal{W}^E_{A\to B}=0$, so

$$\frac{1}{2}\sum_{j,\tilde{j}}\iint\mathcal{W}^k_{A\to B,\epsilon_i^j\epsilon_i^{\tilde{j}}}\mathbb{E}_i\epsilon_i^j\epsilon_i^{\tilde{j}}d\tilde{i}di=0.$$

All put together we have

$$\mathcal{W}^E_{A\to B}\simeq\mathbb{E}\phi_i\Gamma+\int\mathcal{W}^E_{A\to B,\Gamma\Delta_i}\Gamma\Delta_i di.$$

In order for the decomposition to satisfy property (d) it must be the case that $\mathcal{W}^E_{A\to B}\simeq-\mathcal{W}^E_{B\to A}$ which implies (for $\mathcal{W}^E_{B\to A}$ we replace Γ with $-\Gamma$ and Δ_i with $-\Delta_i$)

$$\mathbb{E}\phi_i\Gamma+\int\mathcal{W}^E_{A\to B,\Gamma\Delta_i}\Gamma\Delta_i di\simeq\mathbb{E}\phi_i\Gamma-\int\mathcal{W}^E_{A\to B,\Gamma\Delta_i}\Gamma\Delta_i di$$

and thus $\int\mathcal{W}^E_{A\to B,\Gamma\Delta_i}\Gamma\Delta_i di=0$. We conclude that $\mathcal{W}^E_{A\to B}\simeq\mathbb{E}\phi_i\Gamma$. Identical arguments imply that $\mathcal{W}^R_{A\to B}\simeq\mathbb{E}\phi_i\Delta_i$ and $\mathcal{W}^I_{A\to B}\simeq\mathbb{E}\phi_i\Lambda_i$ as desired.

A.5 Details for Section 3.1.1

We can write the welfare difference as

$$\mathcal{W}^B-\mathcal{W}^A=\mathbb{E}\phi_i\left[\Gamma+\Delta_i+\gamma_i\Lambda_i\right]+R,$$

where R is a residual in decomposition (5). Let $R_\Gamma,R_\Delta,R_\Lambda$ with $R_\Gamma+R_\Delta+R_\Lambda=R$ be parts of R attributed to the aggregate efficiency, redistribution, and insurance components so that the "true" contribution, for example, of aggregate efficiency component is $\frac{\mathbb{E}\phi_i\Gamma+R_\Gamma}{\mathcal{W}^B-\mathcal{W}^A}$. We have

$$\frac{\mathbb{E}\phi_i\Gamma+R_\Gamma}{\mathcal{W}^B-\mathcal{W}^A}=\frac{\mathbb{E}\phi_i\Gamma+R_\Gamma}{\mathbb{E}\phi_i\left[\Gamma+\Delta_i+\gamma_i\Lambda_i\right]+R}=\frac{\mathbb{E}\phi_i\Gamma}{\mathbb{E}\phi_i\left[\Gamma+\Delta_i+\gamma_i\Lambda_i\right]}\times\frac{1+\frac{R_\Gamma}{\mathbb{E}\phi_i\Gamma}}{1+\frac{R}{\mathbb{E}\phi_i\left[\Gamma+\Delta_i+\gamma_i\Lambda_i\right]}}.$$

Therefore, the aggregate efficiency under the true decomposition and the one given in equation (5) coincide if the second term in the second equality is equal to 1 or, equivalently,

$$\frac{R_\Gamma}{R} = \frac{\mathbb{E}\phi_i \Gamma}{\mathbb{E}\phi_i \left[\Gamma + \Delta_i + \gamma_i \Lambda_i\right]}.$$

Analogous arguments apply to the redistribution and insurance components.

The third-order residual in decomposition (5) can be written as

$$R\left(\left(\ln C^Z, \ln w^Z, \mathbf{0}\right), \left(\ln C^B, \ln w^B, \varepsilon^B\right)\right) - R\left(\left(\ln C^Z, \ln w^Z, \mathbf{0}\right), \left(C^A, w^A, \varepsilon^A\right)\right).$$

It converges to zero as $||c^B - c^A|| \to 0$. The speed of this convergence is

$$\max\left\{o\left(||\Gamma^A, \Delta^A, \varepsilon^A||^2\right), o\left(||\Gamma^B, \Delta^B, \varepsilon^B||^2\right)\right\}.$$

Since $\Gamma^B = -\Gamma^A = \frac{1}{2}\Gamma$ and $\Delta_i^B = -\Delta_i^A = \frac{1}{2}\Delta_i$, using the properties of a norm, we have

$$\left\|\Gamma^A, \Delta^A, \varepsilon^A\right\|^2 = \left\|-\frac{1}{2}\Gamma, -\frac{1}{2}\Delta, \varepsilon^A\right\|^2 = \frac{1}{4}\left\|\Gamma, \Delta, -2\varepsilon^A\right\|^2,$$

and $o\left(\frac{1}{4}||\Gamma, \Delta, -2\varepsilon^A||^2\right) = o\left(||\Gamma, \Delta, -2\varepsilon^A||^2\right)$. This and the analogous argument for $||\Gamma^B, \Delta^B, \varepsilon^B||^2$ provide the approximation errors in decomposition (5).

The construction of composition (3.1.1) is identical to that of (5), with appropriate modification of the space X to consist of sequences $\left\{\tilde{\Gamma}, \tilde{\Delta}_i, \tilde{\delta}_i\right\}_i$, but now $\delta_i^B(\xi) = -\delta_i^A(\xi) = \frac{1}{2}\delta_i(\xi)$ and, therefore,

$$\left\|\Gamma^A, \Delta^A, \varepsilon^A\right\|^2 = \left\|\Gamma^B, \Delta^B, \varepsilon^B\right\|^2 = \frac{1}{4}\left\|\Gamma, \Delta, \varepsilon\right\|^2,$$

and $\max\left\{o\left(||\Gamma^A, \Delta^A, \delta^A||^2\right), o\left(||\Gamma^B, \Delta^B, \delta^B||^2\right)\right\} = o\left(\left\|\Gamma, \Delta, \delta\right\|^2\right)$.

A.6 Derivation of equation (13)

As in Section 3.1, we define $\Gamma_k^j \equiv \ln C_k^j - \ln C_k^Z$ and $\Delta_{k,i}^j \equiv \ln w_{k,i}^j - \ln w_{k,i}^Z$, and observe that they satisfy $\Gamma_k^A = -\Gamma_k^B$, $\Delta_{k,i}^A = -\Delta_{k,i}^B$ and, therefore,

$$\Gamma_k^j \Gamma_m^j, \Gamma_k^j \Delta_{m,i}^j, \Delta_{k,i}^j \Delta_{m,i}^j \text{ are t.i.p. for all } k, m, i.$$

We have

$$\mathcal{W}^j(\sigma) = \mathbb{E}\alpha_i U\left(\left\{\exp\left(\sigma\left(\Gamma_k^j + \Delta_{k.i}^j\right)\right)\left(1+\sigma\varepsilon_{k,i}^j\right)c_{k,i}^Z\right\}_k\right).$$

Therefore, its second-order expansion can be written, using the LIE, as

$$\mathcal{W}^j(1) - \mathcal{W}^j(0) \simeq \mathbb{E}\sum_k \alpha_i U_{k,i} c_{k,i}^Z\left(\Gamma^j + \Delta_{k,i}^j\right) + \frac{1}{2}\mathbb{E}\sum_k\sum_m \alpha_i U_{km,i} c_{k,i}^Z c_{m,i}^Z \mathbb{E}_i\left(\varepsilon_{k,i}^j\varepsilon_{m,i}^j\right) + \text{t.i.p.}$$

Since $\mathcal{W}^j = \mathcal{W}^j(1)$, we have

$$\begin{aligned}\mathcal{W}^B - \mathcal{W}^A \quad &\simeq \quad \mathbb{E}\sum_k \alpha_i U_{k,i} c_{k,i}^Z \Gamma_k + \mathbb{E}\sum_k \alpha_i U_{k,i} c_{k,i}^Z \Delta_{k,i} \\ &\quad + \frac{1}{2}\mathbb{E}\sum_k \alpha_i U_{k,i} c_{k,i}^Z \sum_m \frac{U_{km,i} c_{m,i}^Z}{U_{k,i}}\left[\mathbb{E}_i\left(\varepsilon_{k,i}^B \varepsilon_{m,i}^B\right) - \mathbb{E}_i\left(\varepsilon_{k,i}^A \varepsilon_{m,i}^A\right)\right].\end{aligned}$$

Use the approximation

$$cov_i\left(\ln c_{k,i}^j, \ln c_{m,i}^j\right) \simeq \mathbb{E}_i\left(\varepsilon_{k,i}^j \varepsilon_{m,i}^j\right),$$

together with the definitions of $\phi_{k,i}, \gamma_{km,i}$, we obtain (13).

A.7 Proofs for Section 4

A.7.1 Derivation of equation (18)

Floden derives (18) for the case of utilitarian planner. We show here that it holds more generally. We have

$$\begin{aligned}\mathbb{E}\alpha_i U\left(c_i^B\right) \quad &= \quad \mathbb{E}\alpha_i U\left(c_i^{ce,B}\right) = \left(1-p_{redis}^B\right)^{1-\gamma} U\left(C^{ce,B}\right) = \left(1-p_{redis}^B\right)^{1-\gamma}\left(1-p_{insur}^B\right)^{1-\gamma} U\left(C^B\right) \\ &= \quad \left(1-p_{redis}^B\right)^{1-\gamma}\left(1-p_{insur}^B\right)^{1-\gamma}\left(1+\omega_{eff}\right)^{1-\gamma} U\left(C^A\right)\end{aligned}$$

and

$$\mathbb{E}\alpha_i U\left((1+\omega)c_i^A\right) = (1+\omega)^{1-\gamma}\left(1-p_{redis}^A\right)^{1-\gamma}\left(1-p_{insur}^A\right)^{1-\gamma} U\left(C^A\right).$$

Therefore

$$(1+\omega)^{1-\gamma} = \left(1+\omega_{eff}\right)^{1-\gamma}\left(\frac{1-p_{redis}^B}{1-p_{redis}^A}\right)^{1-\gamma}\left(\frac{1-p_{insur}^B}{1-p_{insur}^A}\right)^{1-\gamma},$$

or

$$(1+\omega) = \left(1+\omega_{eff}\right)\left(1+\omega_{redis}\right)\left(1+\omega_{insur}\right).$$

Take logs to get (18).

A.7.2 Derivation of equation (20)

Properties of normal distributions imply that

$$
\begin{aligned}
\mathbb{E}_i \exp\left((1-\tau)\,\xi + \tau\,(1-\tau)\,\frac{\upsilon_\xi^2}{2}\right) &= \mathbb{E}\exp\left((1-\tau)\,e_i + \tau\,(1-\tau)\,\frac{\upsilon_e^2}{2}\right) = 1, \\
\mathbb{E}\alpha_i \ln\left\{\exp\left((1-\tau)\,e_i + \tau\,(1-\tau)\,\frac{\upsilon_e^2}{2}\right)\right\} &= (1-\tau)\mathbb{E}\alpha_i e_i + \tau\,(1-\tau)\,\frac{\upsilon_e^2}{2} \\
&= (1-\tau)\mathrm{cov}(\alpha_i, e_i) - (1-\tau)^2\frac{\upsilon_e^2}{2}, \\
\mathbb{E}\ln\left\{\exp\left((1-\tau)\,\xi_{i,t} + \tau\,(1-\tau)\,\frac{\upsilon_{\xi,t}^2}{2}\right)\right\} &= -\,(1-\tau)^2\,\frac{\upsilon_{\xi,t}^2}{2}.
\end{aligned}
$$

As welfare is given by $\mathcal{W}(\tau) = \mathbb{E}\alpha_i\left\{\ln c_{i,t}(\tau) - \frac{1}{1-\eta}L(\tau)^{1-\eta}\right\}$, substitute (19) and the expressions above into it to obtain (20).

A.7.3 Derivations for equations (21)–(23), and (24)

To derive Benabou-Floden decomposition for the Section 4.2 economy, we begin by noting that

$$
\ln(C(\tau)) - \frac{1}{1+\eta}L(\tau)^{1+\eta} = \frac{1}{1+\eta}\ln(1-\tau) - \frac{1}{1+\eta}(1-\tau).
$$

Thus

$$
\begin{aligned}
\ln\left(1+\omega_{eff}\right) &= \left(\frac{1}{1+\eta}\ln(1-\tau^B) - \frac{1}{1+\eta}(1-\tau^B)\right) - \left(\frac{1}{1+\eta}\ln(1-\tau^A) - \frac{1}{1+\eta}(1-\tau^A)\right) \\
&= \frac{1}{1+\eta}\ln\left(\frac{1-\tau^B}{1-\tau^A}\right) + \frac{1}{1+\eta}\left(\tau^B - \tau^A\right). \qquad (38)
\end{aligned}
$$

Next, we note that $C^{ce}(\tau) = C(\tau)\times\exp\left\{-(1-\tau)^2\frac{\nu_\xi^2}{2}\right\}$, thus $\ln C^{ce}(\tau) - \ln C(\tau) = -(1-\tau)^2\frac{\nu_\xi^2}{2}$, and

$$
\begin{aligned}
\ln\left(1+\omega_{insur}\right) &= \left(\ln C^{ce}(\tau^B) - \ln C(\tau^B)\right) - \left(\ln C^{ce}(\tau^A) - \ln C(\tau^A)\right) \\
&= \frac{\upsilon_\xi^2}{2}\left[\left(1-\tau^A\right)^2 - \left(1-\tau^B\right)^2\right]. \qquad (39)
\end{aligned}
$$

Finally,

$$
\mathcal{W}(\tau) - \left(\ln(C^{ce}(\tau)) - \frac{1}{1+\eta}L(\tau)^{1+\eta}\right) = (1-\tau)cov(\alpha_i, e_i) - (1-\tau)^2\frac{\upsilon_e^2}{2}
$$

so

$$\ln\left(1+\omega_{redis}\right)=\frac{\upsilon_e^2}{2}\left[\left(1-\tau^A\right)^2-\left(1-\tau^B\right)^2\right]+cov\left(\alpha_i,e_i\right)\left[\tau^A-\tau^B\right]. \tag{40}$$

To derive the last lines of (21)–(23), we note that since utility is separable, we can apply our decomposition separately for each good. Consider good c first. From (19) and properties of normal distributions, we have

$$w_{c,i}\left(\xi\right)=\exp\left(\left(1-\tau\right)e_i+\tau\left(1-\tau\right)\frac{\upsilon_e^2}{2}\right),\quad 1+\varepsilon_{c,i}=\exp\left(\left(1-\tau\right)\xi+\tau\left(1-\tau\right)\frac{\upsilon_\xi^2}{2}\right).$$

Therefore,

$$\begin{aligned}
\Gamma_c &= \ln C\left(\tau^B\right)-\ln C\left(\tau^A\right), && (41)\\
\Delta_{c,i} &= \left[\left(1-\tau^B\right)-\left(1-\tau^A\right)\right]e_i+\left[\tau^B\left(1-\tau^B\right)-\tau^A\left(1-\tau^A\right)\right]\frac{\upsilon_e^2}{2}, && (42)\\
\Lambda_{c,i} &= -\left[\left(1-\tau^B\right)^2-\left(1-\tau^A\right)^2\right]\frac{\upsilon_\xi^2}{2}. && (43)
\end{aligned}$$

Given logarithmic preferences and utilitarian weights, we have $\phi_{c,i}=\alpha_i$ and $\gamma_{c,i}=1$ and therefore

$$\begin{aligned}
\text{agg. efficiency}_c &= \ln C\left(\tau^B\right)-\ln C\left(\tau^A\right),\\
\text{redistribution}_c &= \mathbb{E}\alpha_i\Delta_{c,i}=-\left[\left(1-\tau^B\right)^2-\left(1-\tau^A\right)^2\right]\frac{\upsilon_e^2}{2}+cov\left(\alpha_i,e_i\right)\left[\tau^A-\tau^B\right],\\
\text{insurance}_c &= \mathbb{E}\alpha_i\Lambda_{c,i}=-\left[\left(1-\tau^B\right)^2-\left(1-\tau^A\right)^2\right]\frac{\upsilon_\xi^2}{2}.
\end{aligned}$$

We now apply this decomposition to labor. Since there is no heterogeneity in hours, we immediately have $\Delta_{l,i}=\Lambda_{l,i}=0$, $l_i^Z=\sqrt{L\left(\tau^A\right)L\left(\tau^B\right)}=\left[\left(1-\tau^A\right)\left(1-\tau^B\right)\right]^{1/2(1+\eta)}$, and $\phi_{l,i}=\alpha_i\left[\left(1-\tau^A\right)\left(1-\tau^B\right)\right]^{1/2}$. This gives

$$\begin{aligned}
\text{agg. efficiency}_l &= -\sqrt{\left(1-\tau^A\right)\left(1-\tau^B\right)}\left(\ln L\left(\tau^B\right)-\ln L\left(\tau^A\right)\right),\\
\text{redistribution}_l &= \text{insurance}_l=0.
\end{aligned}$$

Combine the decompositions for consumption and labor to get equation the last lines of equations (21)-(23).

Finally, consider the residual term. Let $1+e \equiv \ln \frac{1-\tau^B}{1-\tau^A}$. We have

$$\frac{\left(1-\tau^B\right)-\left(1-\tau^A\right)}{1+\eta} = \frac{1-\tau^A}{1+\eta}\left[\exp e - 1\right] = \frac{1-\tau^A}{1+\eta}\left[e+\frac{1}{2}e^2+O\left(e^3\right)\right]$$

and

$$\sqrt{\left(1-\tau^A\right)\left(1-\tau^B\right)}\frac{\ln\left(1-\tau^B\right)-\ln\left(1-\tau^A\right)}{1+\eta} = \frac{1-\tau^A}{1+\eta}\exp\left(\frac{1}{2}e\right)e = \frac{1-\tau^A}{1+\eta}\left[e+\frac{1}{2}e^2+O\left(e^3\right)\right].$$

This implies that $R = O\left(e^3\right)$.

To derive equation (24) note that

$$1-p^j_{insur} = \frac{C^{ce,j}}{C^j}.$$

Taking logs gives us

$$\ln\left(1-p^j_{insur}\right) = \ln\left(C^{ce,j}\right) - \ln(C^j),$$

so

$$\ln(1+\omega_{insur}) = \left(\ln\left(C^{ce,B}\right)-\ln(C^B)\right) - \left(\ln\left(C^{ce,A}\right)-\ln(C^A)\right).$$

We then have

$$\begin{aligned}
\ln(1+\omega_{insur}) &= \left(\ln\left(C^{ce,B}\right)-\ln(C^B)\right) - \left(\ln\left(C^{ce,A}\right)-\ln(C^A)\right) \\
&\simeq \int \frac{1}{C^Z}c_i^Z\left(\frac{c_i^{ce,B}}{\left(\mathbb{E}c_i^B\right)}-1\right)di - \int \frac{1}{C^Z}c_i^Z\left(\frac{c_i^{ce,A}}{\left(\mathbb{E}c_i^A\right)}-1\right)di \\
&\simeq \int \frac{1}{C^Z}c_i^Z\left(\frac{c_i^{ce,B}}{\left(\mathbb{E}c_i^B\right)}-\frac{c_i^{ce,A}}{\left(\mathbb{E}c_i^A\right)}\right)di.
\end{aligned}$$

So now we have

$$\ln(1+\omega_{insur}) \simeq \int \frac{c_i^Z}{C^Z}\left(\frac{c_i^{ce,B}}{\left(\mathbb{E}c_i^B\right)}-\frac{c_i^{ce,A}}{\left(\mathbb{E}c_i^A\right)}\right)di.$$

A.7.4 Proof of Lemma 1

We focus in the proof only on Floden decomposition, since the proof for Benabou decomposition follows the same steps but is simpler.

As a first step, we want to characterize approximations of consumption certainty equivalent $c_i^{ce,j}$. Define a function $c_i^{ce,j}(\sigma)$ by

$$U\left(c_i^{ce,j}(\sigma)\right) = \mathbb{E}_i U\left(\exp\left(\sigma\left(\Gamma^j + \Delta_i^j\right)\right)\left(1 + \sigma\varepsilon_i^j\right)c_i^Z\right). \tag{44}$$

While $c_i^{ce,j}(1) = c_i^{ce,j}$, it is more convenient to work with an arbitrary σ first. We will prove several intermediate claims first about $c_i^{ce,j}(\sigma)$ and $C_i^{ce,j}(\sigma) = \mathbb{E}c_i^{ce,j}(\sigma)$. Throughout these proofs, unless noted otherwise, U, U_c have arguments $\mathbb{E}c_i^Z$, while $U_{c,i}, U_{cc,i}$ have arguments c_i^Z.

Claim 1. $c_i^{ce,j}(\sigma) = c_i^Z + \sigma c_i^Z\left(\Gamma^j + \Delta_i^j\right) + \frac{\sigma^2}{2}c_i^Z\left[\left(\Gamma^j + \Delta_i^j\right)^2 - var_i\left(\varepsilon_i^j\right)\right] + o\left(\sigma^2\right)$.

Proof. The right and left sides of (44), respectively, are

$$\begin{aligned} RHS(\sigma) &= U_i + \sigma U_{c,i}c_i^Z\left(\Gamma^j + \Delta_i^j\right) + \frac{\sigma^2}{2}U_{cc,i} \times \left(c_i^Z\right)^2\left[\left(\Gamma^j + \Delta_i^j\right)^2 + var_i\left(\varepsilon_i^j\right)\right] \\ &\quad + \frac{\sigma^2}{2}U_{c,i}c_i^Z\left(\Gamma^j + \Delta_i^j\right)^2 + o\left(\sigma^2\right), \end{aligned}$$

and

$$LHS(\sigma) = U\left(\bar{c}_i^{ce,j}\right) + \sigma U_c\left(\bar{c}_i^{ce,j}\right)c_{i,\sigma}^{ce,j} + \frac{\sigma^2}{2}\left[U_{cc}\left(\bar{c}_i^{ce,j}\right)\left(c_{i,\sigma}^{ce,j}\right)^2 + U_c\left(\bar{c}_i^{ce,j}\right) \times c_{i,\sigma\sigma}^{ce,j}\right] + o\left(\sigma^2\right),$$

where $\bar{c}_i^{ce,j} \equiv c_i^{ce,j}(0)$, $c_{i,\sigma}^{ce,j} \equiv \left.\frac{\partial c_i^{ce,j}(\sigma)}{\partial\sigma}\right|_{\sigma=0}$, $c_{i,\sigma\sigma}^{ce,j} \equiv \left.\frac{\partial^2 c_i^{ce,j}(\sigma)}{\partial\sigma^2}\right|_{\sigma=0}$. Since $RHS(\sigma) = LHS(\sigma)$ for all σ, it follows that

$$c_{i,0}^{ce,j} = c_i^Z,\ c_{i,\sigma}^{ce,j} = c_i^Z\left(\Gamma^j + \Delta_i^j\right),\quad c_{i,\sigma\sigma}^{ce,j} = \frac{U_{cc,i}}{U_{c,i}}\left(c_i^Z\right)^2 var_i\left(\varepsilon_i^j\right) + c_i^Z\left(\Gamma^j + \Delta_i^j\right)^2.$$

We have

$$c_i^{ce,j}(\sigma) = c_{i,0}^{ce,j} + \sigma c_{i,\sigma}^{ce,j} + \frac{\sigma^2}{2}c_{i,\sigma\sigma}^{ce,j} + o\left(\sigma^2\right).$$

Substitute previous expression and use the fact that U is CRRA to prove the claim. □

Claim 2. $U\left(C^{ce,j}(\sigma)\right) = \sigma\frac{\mathbb{E}c_i^Z\left(\Gamma^j + \Delta_i^j\right)}{\mathbb{E}c_i^Z} - \frac{\sigma^2}{2}\gamma\frac{\mathbb{E}c_i^Z var_i\left(\varepsilon_i^j\right)}{\mathbb{E}c_i^Z} + t.i.p. + o\left(\sigma^2\right)$.

Proof. By claim 1,

$$\begin{aligned} U\left(C^{ce,j}(\sigma)\right) &= U+\sigma U_c\times \mathbb{E}c_i^Z\left(\Gamma^j+\Delta_i^j\right)+\frac{\sigma^2}{2}U_{cc}\times\left(\mathbb{E}c_i^Z\left(\Gamma^j+\Delta_i^j\right)\right)^2 \\ &\quad +\frac{\sigma^2}{2}U_c\times\left[\mathbb{E}\frac{U_{cc,i}}{U_{c,i}}\left(c_i^Z\right)^2 var_i\left(\varepsilon_i^j\right)+\mathbb{E}c_i^Z\left(\Gamma^j+\Delta_i^j\right)^2\right]+o\left(\sigma^2\right) \\ &= \sigma U_c\mathbb{E}c_i^Z\left(\Gamma^j+\Delta_i^j\right)+\frac{\sigma^2}{2}U_c\mathbb{E}\frac{U_{cc,i}}{U_{c,i}}\left(c_i^Z\right)^2 var_i\left(\varepsilon_i^j\right)+\text{t.i.p.}+o\left(\sigma^2\right). \end{aligned} \tag{45}$$

Use the fact that U is CRRA and evaluate this expression at $\sigma=1$ to prove the claim. □

Claim 3. *Let $x(\sigma)$ be a twice differentiable function, with $\bar{x}=x(0)$, $x_\sigma=x'(0)$, $x_{\sigma\sigma}=x''(0)$, so that*

$$x(\sigma)=\bar{x}+\sigma x_\sigma+\frac{\sigma^2}{2}x_{\sigma\sigma}+o\left(\sigma^2\right).$$

Then

$$\ln x(\sigma)=\ln\bar{x}+\sigma\frac{x_\sigma}{\bar{x}}+\frac{\sigma^2}{2}\left[\frac{x_{\sigma\sigma}}{\bar{x}}-\left(\frac{x_\sigma}{\bar{x}}\right)^2\right]+o\left(\sigma^2\right). \tag{46}$$

Proof. This follows from a routine application of a Taylor expansion of $\ln x(\sigma)$ around $\ln x(0)$. □

Claim 4.

$$\begin{aligned} \ln\mathcal{W}^j &\simeq (1-\gamma)\frac{\mathbb{E}\alpha_i\left(c_i^Z\right)^{1-\gamma}\left(\Gamma^j+\Delta_i^j\right)-\frac{\gamma}{2}\mathbb{E}\alpha_i\left(c_i^Z\right)^{1-\gamma}var_i\left(\varepsilon_i^j\right)}{\mathbb{E}\alpha_i\left(c_i^Z\right)^{1-\gamma}}+t.i.p., \\ \ln U\left(C^{ce,j}\right) &\simeq (1-\gamma)\frac{\mathbb{E}c_i^Z\left(\Gamma^j+\Delta_i^j\right)-\frac{\gamma}{2}\mathbb{E}c_i^Z var_i\left(\varepsilon_i^j\right)}{\mathbb{E}c_i^Z}+t.i.p., \\ \ln U\left(C^j\right) &\simeq (1-\gamma)\frac{\mathbb{E}c_i^Z\left(\Gamma^j+\Delta_i^j\right)}{\mathbb{E}c_i^Z}+t.i.p. \end{aligned}$$

Proof. (*first equation*). From (32), we can write

$$\begin{aligned} \mathcal{W}^j(\sigma) &= \mathcal{W}^Z+\sigma\mathbb{E}\alpha_iU_{c,i}c_i^Z\left(\Gamma^j+\Delta_i^j\right) \\ &\quad +\frac{\sigma^2}{2}\left\{\mathbb{E}\alpha_iU_{c,i}c_i^Z\left(\Gamma^j+\Delta_i^j\right)^2+\mathbb{E}\alpha_iU_{cc,i}\times\left(c_i^Z\right)^2\left[\left(\Gamma^j+\Delta_i^j\right)^2+\left(\varepsilon_i^j\right)^2\right]\right\}+o\left(\sigma^2\right). \end{aligned}$$

Apply claim 3 to get

$$\ln\mathcal{W}^j(\sigma)=\sigma\frac{\mathbb{E}\alpha_iU_{c,i}c_i^Z\left(\Gamma^j+\Delta_i^j\right)}{\mathcal{W}^Z}+\frac{\sigma^2}{2}\frac{\mathbb{E}\alpha_iU_{cc,i}\times\left(c_i^Z\right)^2\left(\varepsilon_i^j\right)^2}{\mathcal{W}^Z}+t.i.p.+o\left(\sigma^2\right).$$

Use the fact that U is CRRA, $\mathcal{W}^Z = (1-\gamma)^{-1} \mathbb{E}\alpha_i \left(c_i^Z\right)^{1-\gamma}$, and that $o\left(\sigma^2\right)$ at $\sigma = 1$ is of order $o\left(\left\|\Gamma^j, \Delta^j, \varepsilon^j\right\|^2\right)$ by equation (29) to show the first equation of the claim.

(*second equation*). Combine claims 2 and 3 to get

$$\ln U\left(C^{ce,j}(\sigma)\right) = \sigma \frac{U_c \times \mathbb{E}c_i^Z\left(\Gamma^j + \Delta_i^j\right)}{U} + \frac{\sigma^2}{2}\frac{U_c \times \left[\mathbb{E}\frac{U_{cc,i}}{U_{c,i}}\left(c_i^Z\right)^2 var_i\left(\varepsilon_i^j\right)\right]}{U} + t.i.p. + o\left(\sigma^2\right).$$

Use the fact that U is CRRA and that $o\left(\sigma^2\right)$ at $\sigma = 1$ is of order $o\left(\left\|\Gamma^j, \Delta^j, \varepsilon^j\right\|^2\right)$ by equation (29), to show the second equation of the claim.

(*third equation*). Let $c_i^j(\sigma) \equiv \exp\left(\sigma\left(\Gamma^j + \Delta_i^j\right)\right)\left(1 + \sigma\varepsilon_i^j\right)c_i^Z$, $C^j(\sigma) \equiv \mathbb{E}c_i^j(\sigma) = \mathbb{E}\exp\left(\sigma\left(\Gamma^j + \Delta_i^j\right)\right)c_i^Z$, where the last equation follows due to the LIE. Therefore,

$$\begin{aligned}
U\left(C^j(\sigma)\right) &= U\left(\mathbb{E}\exp\left(\sigma\left(\Gamma^j + \Delta_i^j\right)\right)c_i^Z\right) \\
&= U + \sigma U_c \mathbb{E}c_i^Z\left(\Gamma^j + \Delta_i^j\right) + \frac{\sigma^2}{2}\left\{U_{cc}\left[\mathbb{E}c_i^Z\left(\Gamma^j + \Delta_i^j\right)\right]^2 + U_c\mathbb{E}\left[c_i^Z\left(\Gamma^j + \Delta_i^j\right)\right]^2\right\} + o\left(\sigma^2\right).
\end{aligned}$$

Apply claim 3 to get

$$\ln U\left(C^j(\sigma)\right) = \sigma\frac{U_c\mathbb{E}c_i^Z\left(\Gamma^j + \Delta_i^j\right)}{U} + t.i.p. + o\left(\sigma^2\right).$$

Use the fact that U is CRRA and that $o\left(\sigma^2\right)$ at $\sigma = 1$ is of order $o\left(\left\|\Gamma^j, \Delta^j, \varepsilon^j\right\|^2\right)$ by equation (29), to show the third equation of the claim. □

Claim 5. $\ln(1+\omega) \simeq \frac{\mathbb{E}\alpha_i\left(c_i^Z\right)^{1-\gamma}(\Gamma+\Delta_i) + \frac{\gamma}{2}\mathbb{E}\alpha_i\left(c_i^Z\right)^{1-\gamma}\left[var_i\left(\varepsilon_i^A\right) - var_i\left(\varepsilon_i^B\right)\right]}{\mathbb{E}\alpha_i\left(c_i^Z\right)^{1-\gamma}}$.

Proof. From equation (17), the term $\ln(1+\omega)$ in Floden decomposition satisfies

$$(1-\gamma)\ln(1+\omega) = \ln\mathcal{W}^B - \ln\mathcal{W}^A. \tag{47}$$

Substitute the first equation from claim 4 to prove this claim. □

Claim 6. $\ln(1+\omega_{insur}) \simeq \frac{\gamma}{2}\frac{\mathbb{E}c_i^Z\left[var_i\left(\varepsilon_i^A\right) - var_i\left(\varepsilon_i^B\right)\right]}{\mathbb{E}c_i^Z}$.

Proof. Using its definition, observe that $\ln(1+\omega_{insur})$ can written as

$$(1-\gamma)\ln(1+\omega_{insur}) = \left[\ln U\left(C^{ce,B}\right) - \ln U\left(C^{ce,A}\right)\right] - \left[\ln U\left(C^B\right) - \ln U\left(C^A\right)\right].$$

Apply the second and third equations from claim 4 and simplify. □

Claim 7. $\ln(1+\omega_{eff}) = \Gamma$.

Proof. This follows from the definitions of $1+\omega_{eff}$ and Γ. □

With these claims we can now prove the second part of the lemma. Suppose that condition (25) is satisfied. Then $2\left[var_i\left(\varepsilon_i^A\right) - var_i\left(\varepsilon_i^B\right)\right] \simeq \Lambda$ for all i and therefore

$$\begin{aligned}
\frac{\ln(1+\omega_{insur})}{\ln(1+\omega)} &= \frac{\gamma\Lambda\mathbb{E}\alpha_i\left(c_i^Z\right)^{1-\gamma}}{\mathbb{E}\alpha_i\left(c_i^Z\right)^{1-\gamma}(\Gamma+\Delta_i+\gamma\Lambda)} + o(1), \\
\frac{\ln(1+\omega_{eff})}{\ln(1+\omega)} &= \frac{\Gamma\mathbb{E}\alpha_i\left(c_i^Z\right)^{1-\gamma}}{\mathbb{E}\alpha_i\left(c_i^Z\right)^{1-\gamma}(\Gamma+\Delta_i+\gamma\Lambda)} + o(1),
\end{aligned}$$

and, since equation (18) holds,

$$\frac{\ln(1+\omega_{redis})}{\ln(1+\omega)} = \frac{\mathbb{E}\alpha_i\left(c_i^Z\right)^{1-\gamma}\Delta_i}{\mathbb{E}\alpha_i\left(c_i^Z\right)^{1-\gamma}(\Gamma+\Delta_i+\gamma\Lambda)} + o(1).$$

The first terms on the right sides of these equations are the very same terms that we obtained using our decomposition (6) under the assumptions of the lemma. Thus, the two decompositions coincide up to $o(1)$, meaning that $o(1) \to 0$ as $\|\Gamma, \Delta, \varepsilon\| \to 0$. Since our decomposition satisfies Properties (a), (b), and (c), so does Floden's, to the order $o(1)$.

A.7.5 More details for example referenced in Footnote (9)

Condition (25) relies critically on the assumption that households hold no assets. Since asset dispersion plays an important role in many heterogeneous households economies (including the one considered by Floden (2001)), our decomposition will differ from that of Benabou or Floden's approach more broadly. To illustrates that those differences can be quite large, we consider the following variant of the Section 4.2 Benabou economy.

We maintain the assumption that U is logarithmic, so that the decompositions of Benabou and Floden coincide, and that the planner is utilitarian. We assume that consumptions of households under policies A and B are

$$\begin{aligned}
c_i^A &= a_i + w_i(1+\tilde{\varepsilon}_i), \\
c_i^B &= a_i + w_i,
\end{aligned}$$

where $\{a_i, w_i\}_i$ are some non-stochastic variables and $\tilde{\varepsilon}_i$ is a non-trivial stochastic process. We assume that $\mathbb{E}_i\tilde{\varepsilon}_i = 0$ and that $var_i(\tilde{\varepsilon}_i) = var(\tilde{\varepsilon})$ for all i. This highly stylized example captures key features of richer heterogeneous households models like Aiyagari (1995) in which household consumption comes from asset income (captured here by a_i) and labor income (w_i), which is also subject to idiosyncratic shocks ($\tilde{\varepsilon}_i$ here). Policy B is a social insurance program that removes all the uncertainty that households face about their earnings without reallocating resources across households or changing the aggregate amount of resources. Since households are risk-averse, policy B improves welfare. All improvement comes from better insurance.[22]

In this reform, decomposition (6) attributes 100% of welfare gains to the insurance component. We now apply the Benabou-Floden decomposition. Since it is not available in closed form for this example, we consider approximations of their expressions for small values of idiosyncratic shocks, $||\varepsilon||$. Certainty equivalents of consumption for household i are

$$\begin{aligned} c_i^{ce,A} &= c_i^Z - \frac{1}{2}\bar{c}_i^Z \left(\frac{w_i}{a_i + w_i}\right)^2 var\left(\varepsilon\right) + o\left(||\varepsilon||^2\right), \\ c_i^{ce,B} &= c_i^Z \ , \end{aligned} \tag{48}$$

where $c_i^Z = a_i + w_i$. Certainty equivalents of consumption under policy B coincide with expected consumption c_i^Z, since households face no uncertainty. Under policy A, certainty equivalents equal expected consumptions adjusted by the coefficient of risk aversion (which is equal to one with logarithmic preferences) and the variance of consumption $\left(\frac{w_i}{a_i+l_i}\right)^2 var\left(\tilde{\varepsilon}\right)$. Note that even though all households face the same uncertainty about labor earnings $\ln\left[w_i\left(1+\tilde{\varepsilon}_i\right)\right]$, their consumption risk varies due to heterogeneity in their asset holdings.

Using expressions (48), it is easy to show that

$$\begin{aligned} \ln\left(1+\omega_{insur}\right) &= \ln \mathbb{E} c_i^{ce,B} - \ln \mathbb{E} c_i^{ce,A} = \frac{1}{2} var\left(\tilde{\varepsilon}\right) \frac{\mathbb{E} c_i^Z \left(\frac{w_i}{a_i+w_i}\right)^2}{\mathbb{E} c_i^Z} + o\left(||\varepsilon||^2\right), \\ \ln\left(1+\omega\right) &= \mathbb{E} \ln c_i^{ce,B} - \mathbb{E} \ln c_i^{ce,A} = \frac{1}{2} var\left(\tilde{\varepsilon}\right) \mathbb{E}\left(\frac{w_i}{a_i + w_i}\right)^2 + o\left(||\varepsilon||^2\right). \end{aligned}$$

[22] The assumed consumption policy and the reform can be microfounded as a special case of the Section 4.2, with $w_i = \exp(e_i)$ and $1 + \tilde{\varepsilon}_i = \exp(\xi_i)$, in which the labor supply elasticity parameter $\eta \to \infty$ and the tax function parameters are allowed to condition on ex-ante heterogeneity e_i.

Combining these two expressions, we get that

$$\frac{\ln\left(1+\omega_{insur}\right)}{\ln\left(1+\omega\right)} = \frac{\mathbb{E}c_i^Z \left(\frac{w_i}{a_i+w_i}\right)^2}{\mathbb{E}c_i^Z \times \mathbb{E}\left(\frac{w_i}{a_i+w_i}\right)^2} + o\left(1\right), \tag{49}$$

where $o\left(1\right) \to 0$ as $||\varepsilon|| \to 0$.

Equation (49) shows that the Benabou-Floden decomposition would assign 100% of welfare gains to insurance only if $c_i^Z = a_i + w_i$ is uncorrelated with $\left(\frac{w_i}{a_i+w_i}\right)^2$ across households. This condition is generally not satisfied unless $a_i = 0$ for all i, so that households have no asset income.

The insurance component in equation (49) is always positive, but it is easy to see that it can be arbitrarily large or small. Suppose that the relationship between a_i and l_i is

$$a_i + l_i = l_i^{\kappa} \text{ for all } i,$$

where the parameter κ captures covariance between assets and labor income. Furthermore, assume that l_i is distributed according to a Pareto distribution with shape parameter ρ. In that case expression (49) implies that

$$\frac{\ln\left(1+\omega_{insur}\right)}{\ln\left(1+\omega\right)} = \frac{\left(\rho-\kappa\right)\left(\rho-\left(2-2\kappa\right)\right)}{\rho\left(\rho-\left(2-\kappa\right)\right)} + o\left(1\right).$$

The first term on the right hand side is well defined so long as $\rho > \max\left\{2-\kappa, \kappa, 2-2\kappa\right\}$. By varying ρ and κ, the left side of the above equation can take any value in the $(0,\infty)$ interval. The residual term $o\left(1\right)$ can be made arbitrarily small by choosing small enough idiosyncratic shocks. This implies that the Benabou-Floden decomposition could assign any value in $(0,\infty)$ to the insurance component in our simple social insurance example. For instance, set $\rho = 3$ (which is in the range of estimates for the Pareto exponents of US labor incomes reported in de Vries and Toda (2022)). At $\kappa = 0$, asset heterogeneity fully offsets labor income heterogeneity to make c_i^Z same across households. Thus, there is no redistribution induced by the policy, and in this case, the Benabou-Floden decomposition assigns 100% to insurance. For a sufficiently large κ, assets are positively correlated with labor income. Wealth-rich households value the reduction in labor income risk less than wealth-poor households. Benabou-Floden decomposition views this as redistribution and assigns a share smaller than 100% to insurance. Since aggregate consumption C is unchanged, and the three components always sum to one, this also implies that the Benabou-Floden decomposition could assign any value in $(-\infty, 1)$ to the redistribution component. We summarize this analysis in

Claim 8. *In the constructed example, depending on κ, ρ, and the shock process ε, the insurance share $\frac{\ln(1+\omega_{insur})}{\ln(1+\omega)}$ can take any value in $(0, \infty)$, and the redistribution share $\frac{\ln(1+\omega_{redis})}{\ln(1+\omega)}$ can take any value in $(-\infty, 1)$.*

A.7.6 Benabou-Floden decomposition violate Properties (a),(b), and (c).

We focus in the proof only on Floden decomposition, since the proof for Benabou decomposition follows the same steps but is simpler.

It is easy to see that generically Properties (a), (b), and (c) will be violated in the Floden decomposition. Consider, for example, Property (a). Take any allocation $\left\{c_i^A\right\}_i$, where consumptions of households are non-trivial stochastic process, and construct $\left\{c_i^B\right\}_i$ by $c_i^B = Dc_i^A$ for all i for some $D > 0$. Property (a) is satisfied if all welfare changes from this policy are attributed to the aggregate efficiency component. This would require that $1 - p_{insur}^B = 1 - p_{insur}^A$. Since $C^B = DC^A$, this would be the case only if $C^{ce,B} = DC^{ce,A}$. But for arbitrary U function, there is no reason to expect that $\mathbb{E}c_i^{ce}$ scales with D under policy B. On the other hand, if U is CRRA, using equation (7) it is easy to verify that $c_i^{ce,B} = Dc_i^{ce,A}$ for all i, and therefore $C^B = DC^A$. Failure of other properties follow from analogous arguments.

A.7.7 Davila and Schaab's decomposition violates Properties (a), (b), and (c).

We show that Davila and Schaab (2022) decomposition violates property (a)–(c) and disagrees with Benabou (2002) (and Floden (2001)) by considering the following examples

Property (a) Consider allocations in which the policy scales all households consumption by a common factor $\exp\{\theta\}$, that is, $c_i(\theta, \xi) = (\exp\{\theta\} C) \times (1 + \epsilon_i(\xi)) \times w_i$, with $\int w_i di = 1$ and $\mathbb{E}_i \epsilon_i = 0$. By construction, this policy affects neither shocks $\{\epsilon_i\}$ nor ex-ante consumption shares $\{w_i\}$. It is easy to verify that $\partial_\theta \ln \mathbb{E}c_i(\theta) = 1$ and $\partial_\theta \ln\left(\frac{\mathbb{E}_i c_i(\theta)}{\mathbb{E}c_i(\theta)}\right) = \partial_\theta \ln\left(c_i(\xi, \theta) / \mathbb{E}_i c_i(\theta)\right) = 0$. So decomposition (12) assigns 100% of welfare gains to aggregate efficiency. Now apply the Davila and Schaab decomposition. From equation (26) we obtain that the redistribution (i.e., the third) term is

$$cov\left(\frac{w_i^{-\gamma}\alpha_i \int \Pr(d\xi)\left[1+\epsilon_i(\xi)\right]^{-\gamma}}{\int\int w_i^{-\gamma}\alpha_i \Pr(d\xi)\left[1+\epsilon_i(\xi)\right]^{-\gamma} di}, \frac{w_i \int \Pr(d\xi)\left[1+\epsilon_i(\xi)\right]^{1-\gamma}}{\int \Pr(d\xi)\left[1+\epsilon_i(\xi)\right]^{-\gamma}}\right), \tag{50}$$

where cov is the cross-sectional covariance. The covariance term in (50) does not generally equal zero. For instance, set $\epsilon_i(s) = 0$ and $\gamma = 1$, then the expression (50) equals $cov\left(\frac{\alpha_i}{w_i}, w_i\right)$ and will be non-zero except in pathological cases.

Notice that when $\epsilon_i(s) = 0$ and $\gamma = 1$, our decomposition (12) agrees with Benabou (2002) (and Floden (2001)). This shows that the Davila and Schaab (2022) decomposition is inconsistent with the Benabou and Floden decompositions too.

Property (b) Consider an allocation under policy θ given by $c_i(\xi, \theta) = C \times (1 + \epsilon_i(\xi)) \times (1 + \theta x_i)$ with $\int x_i di = 0$ and $\int \Pr(d\xi)\, \epsilon_i(\xi) = 0$. It is easy to see decomposition (12) will attribute welfare gains from any change in θ to redistribution. Now apply (26). The aggregate efficiency (first) term is

$$C \int \Pr(d\xi) \left[\left\{ \int (1 + \epsilon_i(\xi))\, x_i di \right\} \times \left\{ \int \left(\frac{[1 + \theta x_i(\xi)]^{-\gamma}}{\int \Pr(d\xi)\, [1 + \theta x_i(\xi)]^{-\gamma}} \right) di \right\} \right].$$

Observe that when $cov(\epsilon_i(\xi), x_i) \neq 0$, Davila and Schaab's decomposition will attribute a nonzero contribution to aggregate efficiency.

Property (c) Consider an allocation under a policy θ such that $c_i(\theta, \xi) = C \times (1 + \theta x_i(\xi)) \times w_i$ with $\int w_i di = 1$ and $\int \Pr(d\xi)\, x_i(\xi) = 0$. Evidently, $\partial_\theta \ln \mathbb{E} c_i(\theta) = \partial_\theta \ln \left(\frac{\mathbb{E}_i c_i(\theta)}{\mathbb{E} c_i(\theta)} \right) = 0$ so decomposition (12) attributes all welfare gains from re-scaling risk to insurance with both aggregate efficiency and redistribution being zero. Now apply (26). The aggregate efficiency (first) term is

$$C \int \left[\left\{ \int w_i x_i(\xi)\, di \right\} \times \left\{ \int \left(\frac{[1 + \theta x_i(\xi)]^{-\gamma}}{\int \Pr(d\xi)\, [1 + \theta x_i(\xi)]^{-\gamma}} \right) di \right\} \right] \Pr(d\xi)$$

This term is zero when $\theta = 0$ and the derivative with respect to θ at $\theta = 0$ is

$$\begin{aligned} &\frac{d}{d\theta} \int \Pr(d\xi) \left[\left\{ \int w_i x_i(\xi)\, di \right\} \times \left\{ \int \left(\frac{[1 + \theta x_i(\xi)]^{-\gamma}}{\int \Pr(d\xi)\, [1 + \theta x_i(\xi)]^{-\gamma}} \right) di \right\} \right] \\ &= -\gamma C \int \Pr(d\xi) \left[\left(\int w_i x_i(\xi)\, di \right) \times \left(\int x_i(\xi)\, di \right) \right] \end{aligned}$$

This derivative is generally nonzero. We conclude that more generally gains from a change in θ that alters only ex post risk but that keeps fixed both ex ante shares and aggregate consumption ends up affecting Davila and Schaab's aggregate efficiency component in their decomposition.

A.7.8 Example to show why efficiency and insurance components, and not just the redistribution component, should depend on the Pareto weights.

The next example spotlights another difference between our decomposition and Davila and Schaab's. From the definition of $\hat{\phi}_i$, we observe that Pareto weights appear in all three components of equation (12), while they only appear in the redistribution (third) term in decomposition (26). We next consider an example that brings out the economic reasons for why efficiency and insurance components, and not just the redistribution component, should depend on the Pareto weights.

Let individuals be of ex ante types 1 and 2. We set index $i \in [0, .5)$ for type 1 and index $i \in [0.5, 1]$ for type 2 so that both types have measure $\frac{1}{2}$. The baseline allocation is

$$c_i\left(\theta_E, \theta_I, \xi\right) = \theta_E \quad i \in [0, .5),$$

$$c_i\left(\theta_E, \theta_I, \xi\right) = 1 + \theta_I x_i\left(\xi\right) \quad i \in [0.5, 1],$$

where we assume that $x_i\left(\xi\right)$ is identically and independently distributed across $i \in [0.5, 1]$ with $\int_{0.5}^{1} x_i\left(\xi\right) di = 0$, and the vector $\theta = \left(\theta_E, \theta_I\right)$ indexes policies. The component θ_E affects the level of consumption for type 1 households. The component θ_I of the policy affects the degree of partial insurance (risk-sharing) for type 2 households. A planner assigns Pareto weights

$$\alpha_i = 2\alpha, \quad i \in [0, .5)$$

$$\alpha_i = 2 - 2\alpha, \quad i \in [0.5, 1]$$

Now consider the welfare effects and the decomposition thereof induced by changing policy parameters $\left(\theta_E, \theta_I\right)$ one at a time. These policies affect only a subset of households, and therefore, aggregate welfare gains (or losses) should automatically be larger when the welfare weight of affected households is bigger. Through the lens of decomposition (12), this implies that all components (and not just redistribution) will vary with Pareto weights.

To see this, set $\theta_E = 1$ and vary θ_I. A lower (higher) θ_I holds the total endowment constant, provides no redistribution either across types or within types before idiosyncratic risk is realized, and reflects only better (worse) risk sharing. Observe that for any α, changing θ_I changes the level of partial insurance only for a type 2 person and does not affect type 1 people. So it is natural to expect that the share of gains coming from insurance is 100%, and that as full weight is placed on type 1 households, that is, $\alpha \to 1$, total welfare gains, and therefore gains from insurance should both approach zero. Apply decomposition (12) to see that $\partial_{\theta_I} ln\left(c_i\left(1,\theta_I,s\right)/\mathbb{E}_i c_i\right)$ equals zero for $i\in[0,0.5)$ and that it equals $\frac{x_i(s)}{1+\theta_I x_i(s)}$ for $i\in[0.5,1)$. So

$$\partial_{\theta_I}\mathcal{W}\left(1,\theta_I\right)=2\left(1-\alpha\right)\int_{0.5}^{1}\int \Pr\left(d\xi\right)U_{c,i}\left(1+\theta_I x_i\left(\xi\right)\right)x_i\left(\xi\right)di,$$

which confirms our anticipation.

Now apply Davila and Schaab's decomposition. It is clear that their aggregate efficiency and risk-sharing (first and second) components are invariant to changes in α, so their decomposition attributes changes in partial insurance to a combination of risk sharing and redistribution. For this example, the aggregate efficiency (first) term is zero, but the risk-sharing is not, and as $\alpha\to 1$, the risk-sharing (second) term equals the negative of redistribution (third) term.

Next, consider a reform that changes θ_E and keeps $\theta_I=0$ fixed. By changing the consumption of type 1 households, this reform changes the size of the pie $(1+\theta_E\ )$. Our decomposition (12) implies that the contribution from aggregate efficiency is increasing in the parameter α that increases the weight on type 1 households and that it approaches 100% as $\alpha\to 1$. On the other hand, from equation (26), the aggregate efficiency term for Davila and Schaab's decomposition equals $\frac{1}{2}$ for all values of α and the share of aggregate efficiency approaches 50% as $\alpha\to$1.[23]

A.7.9 Details of applying marginal decompositions to Section 4.2 economy

In this section, we apply marginal versions of ours and Benabou's, and Davila and Schaab decomposition to the Section 4.2 economy.

Marginal decomposition for Benabou Decomposition Benabou-Floden decomposition is defined for two economies A and B. We index economy A with a policy vector τ and economy B with a policy vector $\tau+d\tau$. Next we define shares of each component as $d\tau\to 0$. Start with Benabou's decomposition (15), we can define shares in a marginal decomposition as

$$\partial_\tau W^{BF,agg.eff}=\lim_{d\tau\to 0}\frac{U\left(C\left(\tau\right)\right)-U\left(C\left(\tau+d\tau\right)\right)}{\mathcal{W}\left(\tau\right)-\mathcal{W}\left(\tau+d\tau\right)}$$

$$\partial_\tau W^{BF,redist}=\lim_{d\tau\to 0}\frac{\left\{\mathcal{W}\left(\tau\right)-\mathcal{W}\left(\tau+d\tau\right)\right\}-\left\{U\left(C^{ce}\left(\tau\right)\right)-U\left(C^{ce}\left(\tau+d\tau\right)\right)\right\}}{\mathcal{W}\left(\tau\right)-\mathcal{W}\left(\tau+d\tau\right)}$$

[23]In fact, their decomposition implies the redistribution term to be zero only when Pareto weights are utilitarian or when $\alpha=\frac{1}{2}$.

$$\partial_\tau W^{BF,ins} = \lim_{d\tau\to 0} \frac{\{U\left(C^{ce}\left(\tau\right)\right) - U\left(C^{ce}\left(\tau + d\tau\right)\right)\} - U\left(C\left(\tau\right)\right) - U\left(C\left(\tau + d\tau\right)\right)}{\mathcal{W}\left(\tau\right) - \mathcal{W}\left(\tau + d\tau\right)}$$

Now we apply the Benabou marginal decomposition to Section 4.2 economy. First, we approximate $\mathcal{W}\left(\tau\right) - \mathcal{W}\left(\tau + d\tau\right)$ for small $d\tau$. Use the expression for $C(\tau) = L(\tau) = (1 - \tau)^{\frac{1}{1+\eta}}$, and equation (20), which is the expression for $\mathcal{W}\left(\tau\right)$ to get

$$\mathcal{W}\left(\tau\right) - \mathcal{W}\left(\tau + d\tau\right) = \left[-\frac{\tau}{(1+\eta)(1-\tau)} + \left(1-\tau\right)\nu_e^2 - cov\left(\alpha_i, e_i\right) + \left(1-\tau\right)\upsilon_\xi^2\right] d\tau + o\left(d\tau\right).$$

Next we use expressions (38), (39), and (40) for each of the components and set $\tau^A = \tau$, $\tau^B = \tau + d\tau$, and take limits with $d\tau$ to get

$$\partial_\tau W^{BF,agg.eff} = \frac{-\frac{\tau}{(1+\eta)(1-\tau)}}{-\frac{\tau}{(1+\eta)(1-\tau)} + \left(1-\tau\right)\nu_e^2 - cov\left(\alpha_i, e_i\right) + \left(1-\tau\right)\upsilon_\xi^2},$$

$$\partial_\tau W^{BF,Ins} = \frac{\left(1-\tau\right)\upsilon_\xi^2}{-\frac{\tau}{(1+\eta)(1-\tau)} + \left(1-\tau\right)\nu_e^2 - cov\left(\alpha_i, e_i\right) + \left(1-\tau\right)\upsilon_\xi^2},$$

and

$$\partial_\tau W^{BF,redist} = \frac{\left(1-\tau\right)\nu_e^2 - cov\left(\alpha_i, e_i\right)}{-\frac{\tau}{(1+\eta)(1-\tau)} + \left(1-\tau\right)\nu_e^2 - cov\left(\alpha_i, e_i\right) + \left(1-\tau\right)\upsilon_\xi^2}.$$

Let $\partial_\tau W^{BEGS,agg.eff}, \partial_\tau W^{BEGS,Ins}, \partial_\tau W^{BEGS,redist}$ be shares for our marginal decomposition defined in (12). For the Section 4.2 economy, $\hat{\phi}_i\left(\tau, \xi\right) = \alpha_i$ and $\gamma_i = 1$ and using expressions (41)-(43), we get that $\partial_\tau W^{BF,agg.eff} = \partial_\tau W^{BEGS,agg.eff}$, $\partial_\tau W^{BF,redist} = \partial_\tau W^{BEGS,redist}$, and $\partial_\tau W^{BF,ins} = \partial_\tau W^{BEGS,ins}$.

We next show that Davila and Schaab decomposition disagrees with Benabou/our decomposition in a special case of the Section 4.2 economy. Assume that $\alpha_i = 1$, and $\upsilon_\xi^2 = 0$.

Davila and Schaab efficiency component To apply the Davila and Schaab decomposition we first need to construct a consumption equivalent change in policy since the policy effects both labor and consumption.

$$\frac{du_{i|c}}{d\tau} = \frac{dc_i}{d\tau} - \frac{l_i(\tau)^\eta}{c_i(\tau)^{-1}}\frac{dl_i}{d\tau}.$$

This will take the place of $\frac{dc_i}{d\tau}$ in (26). In the main text we show taht

$$c_i\left(e_i, \xi_i, \tau\right) = C\left(\tau\right) \times \exp\left(\left(1-\tau\right)e_i + \tau\left(1-\tau\right)\frac{\upsilon_e^2}{2}\right) \times \exp\left(\left(1-\tau\right)\xi_i + \tau\left(1-\tau\right)\frac{\upsilon_\xi^2}{2}\right). \tag{51}$$

From (51), we note that

$$\frac{dc_i}{d\tau} = \left(-\frac{1}{(1+\eta)(1-\tau)} - e_i + (1-2\tau)\frac{\nu_e^2}{2} - \xi_i + (1-2\tau)\frac{\nu_\xi^2}{2}\right) c_i(\tau)$$
$$\frac{l_i(\tau)^\eta}{c_i(\tau)^{-1}} = (1-\tau)^{\frac{\eta+1}{1+\eta}} \times \exp\left((1-\tau)e_i + \tau(1-\tau)\frac{\nu_e^2}{2}\right) \times \exp\left((1-\tau)\xi_i + \tau(1-\tau)\frac{\nu_\xi^2}{2}\right)$$
$$\frac{dl_i}{d\tau} = -\frac{1}{(1+\eta)(1-\tau)} l_i(\tau)$$

Consider the case where $\tau > 0$ but $\nu_\xi^2 = 0$. Then we have that

$$\begin{aligned}\frac{du_{i|c}}{d\tau} &= \left(-\frac{1}{(1+\eta)(1-\tau)} - e_i + (1-2\tau)\frac{\nu_e^2}{2}\right) c_i(\tau) + \frac{1}{(1+\eta)} c_i(\tau) \\ &= \left(-\frac{\tau}{(1+\eta)(1-\tau)} - e_i + (1-2\tau)\frac{\nu_e^2}{2}\right) c_i(\tau).\end{aligned}$$

According to (26), the aggregate efficiency term is

$$\int \frac{du_{i|c}}{d\tau} di = \left(-\frac{\tau}{(1+\eta)(1-\tau)} + (1-2\tau)\frac{\nu_e^2}{2}\right) \int c_i(\tau) di - \int e_i c_i(\tau) di.$$

We know that $\int c_i(\tau)di = (1-\tau)^{\frac{1}{1+\eta}}$. We can write that second term as

$$\begin{aligned}\int e_i c_i(\tau) di &= (1-\tau)^{\frac{1}{1+\eta}} \int e_i \exp\left((1-\tau)e_i + \tau(1-\tau)\frac{\nu_e^2}{2}\right) di \\ &= (1-\tau)^{\frac{1}{1+\eta}} \frac{1}{(1-\tau)} \int (1-\tau)e_i \exp\left((1-\tau)e_i + \tau(1-\tau)\frac{\nu_e^2}{2}\right) di \\ &= (1-\tau)^{\frac{1}{1+\eta}} \frac{1}{(1-\tau)} \int \hat{e}_i \exp\left(\hat{e}_i\right) di \\ &\quad - (1-\tau)^{\frac{1}{1+\eta}} \frac{1}{(1-\tau)} \int \tau(1-\tau)\frac{\nu_e^2}{2} \exp\left((1-\tau)e_i + \tau(1-\tau)\frac{\nu_e^2}{2}\right) di \\ &= (1-\tau)^{\frac{1}{1+\eta}} \frac{1}{(1-\tau)} \frac{(1-\tau)^2\nu_e^2}{2} - (1-\tau)^{\frac{1}{1+\eta}} \tau \frac{\nu_e^2}{2} \\ &= (1-\tau)^{\frac{1}{1+\eta}} (1-2\tau) \frac{\nu_e^2}{2}\end{aligned}$$

where $\hat{e}_i = (1-\tau)e_i + \tau(1-\tau)\frac{\nu_e^2}{2}$. So we get that

$$\int \frac{du_{i|c}}{d\tau} di = -\frac{\tau}{(1+\eta)(1-\tau)} (1-\tau)^{\frac{1}{1+\eta}}.$$

Davila and Schaab redistribution component Redistribution is given by

$$cov\left(\frac{1/c_i(\tau)}{\int 1/c_i(\tau)di}, \frac{du_{i|c}}{d\tau}\right) = \left(\int 1/c_i(\tau)di\right)^{-1}\left(\int 1/c_i(\tau)\frac{du_{i|c}}{d\tau}di - \left(\int 1/c_i(\tau)di\right)\int \frac{du_{i|c}}{d\tau}di\right)$$

To simplify this, first note that

$$\begin{aligned}
\left(\int 1/c_i(\tau)di\right)^{-1} &= \left((1-\tau)^{\frac{-1}{1+\eta}}\int \exp\left(-(1-\tau)e_i - \tau(1-\tau)\frac{\nu_e^2}{2}\right)di\right)^{-1} \\
&= (1-\tau)^{\frac{1}{1+\eta}}\exp\left(\tau(1-\tau)\frac{\nu_e^2}{2}\right)\left(\int \exp\left(-(1-\tau)e_i\right)di\right)^{-1} \\
&= (1-\tau)^{\frac{1}{1+\eta}}\exp\left(\tau(1-\tau)\frac{\nu_e^2}{2}\right)\left(\exp\left((1-\tau)\frac{\nu_e^2}{2} + (1-\tau)^2\frac{\nu_e^2}{2}\right)\right)^{-1} \\
&= (1-\tau)^{\frac{1}{1+\eta}}\exp\left(-(1-\tau)^2\nu_e^2\right)
\end{aligned}$$

and

$$\begin{aligned}
\int 1/c_i(\tau)\frac{du_{i|c}}{d\tau}di &= \int\left(-\frac{\tau}{(1+\eta)(1-\tau)} - e_i + (1-2\tau)\frac{\nu_e^2}{2}\right)di \\
&= -\frac{\tau}{(1+\eta)(1-\tau)} + (1-\tau)\nu_e^2
\end{aligned}$$

This implies that

$$\begin{aligned}
\left(\int 1/c_i(\tau)di\right)\int \frac{du_{i|c}}{d\tau}di &= (1-\tau)^{\frac{-1}{1+\eta}}\exp\left((1-\tau)^2\nu_e^2\right)\frac{\tau}{(1+\eta)(1-\tau)}(1-\tau)^{\frac{1}{1+\eta}} \\
&= \exp\left((1-\tau)^2\nu_e^2\right)\frac{\tau}{(1+\eta)(1-\tau)}
\end{aligned}$$

All combined we have that the redistribution term is given by

$$cov\left(\frac{1/c_i(\tau)}{\int 1/c_i(\tau)di}, \frac{du_{i|c}}{d\tau}\right) = (1-\tau)^{\frac{1}{1+\eta}}\exp\left(-(1-\tau)^2\nu_e^2\right)\left(\left(\exp\left((1-\tau)^2\nu_e^2\right)-1\right)\frac{\tau}{(1+\eta)(1-\tau)} + (1-\tau)\nu_e^2\right)$$

Davila and Schaab shares The Davila and Schaab component shares are given by

$$\partial_\tau W^{DS,agg.eff} = \frac{-\frac{\tau}{(1+\eta)(1-\tau)}(1-\tau)^{\frac{1}{1+\eta}}}{-\frac{\tau}{(1+\eta)(1-\tau)}(1-\tau)^{\frac{1}{1+\eta}} + (1-\tau)^{\frac{1}{1+\eta}}\exp\left(-(1-\tau)^2\nu_e^2\right)\left(\left(\exp\left((1-\tau)^2\nu_e^2\right)-1\right)\frac{\tau}{(1+\eta)(1-\tau)} + (1-\tau)\nu_e^2\right)}$$

$$\partial_\tau W^{DS,redist} = \frac{(1-\tau)^{\frac{1}{1+\eta}}\exp\left(-(1-\tau)^2\nu_e^2\right)\left(\left(\exp\left((1-\tau)^2\nu_e^2\right)-1\right)\frac{\tau}{(1+\eta)(1-\tau)} + (1-\tau)\nu_e^2\right)}{-\frac{\tau}{(1+\eta)(1-\tau)}(1-\tau)^{\frac{1}{1+\eta}} + (1-\tau)^{\frac{1}{1+\eta}}\exp\left(-(1-\tau)^2\nu_e^2\right)\left(\left(\exp\left((1-\tau)^2\nu_e^2\right)-1\right)\frac{\tau}{(1+\eta)(1-\tau)} + (1-\tau)\nu_e^2\right)}$$

To compare them to Benabou/our marginal shares, we use a small ν_e^2 approximation. This gives us

$$cov\left(\frac{1/c_i(\tau)}{\int 1/c_i(\tau)di}, \frac{du_{i|c}}{d\tau}\right) \approx (1-\tau)^{\frac{1}{1+\eta}}(1-\tau)\nu_e^2\left(1+\frac{\tau}{(1+\eta)(1-\tau)}\right).$$

and the shares are given by

$$\partial_\tau W^{DS,agg.eff} \approx \frac{-\frac{\tau}{(1+\eta)(1-\tau)}(1-\tau)^{\frac{1}{1+\eta}}}{-\frac{\tau}{(1+\eta)(1-\tau)}(1-\tau)^{\frac{1}{1+\eta}} + (1-\tau)^{\frac{1}{1+\eta}}(1-\tau)\nu_e^2\left(1+\frac{\tau}{(1+\eta)(1-\tau)}\right)}$$

$$\partial_\tau W^{DS,redist} \approx \frac{(1-\tau)^{\frac{1}{1+\eta}}(1-\tau)\nu_e^2\left(1+\frac{\tau}{(1+\eta)(1-\tau)}\right)}{-\frac{\tau}{(1+\eta)(1-\tau)}(1-\tau)^{\frac{1}{1+\eta}} + (1-\tau)^{\frac{1}{1+\eta}}(1-\tau)\nu_e^2\left(1+\frac{\tau}{(1+\eta)(1-\tau)}\right)}$$

These simplify to

$$\partial_\tau W^{DS,agg.eff} \approx \frac{-\frac{\tau}{(1+\eta)(1-\tau)}}{-\frac{\tau}{(1+\eta)(1-\tau)} + (1-\tau)\nu_e^2\left(1+\frac{\tau}{(1+\eta)(1-\tau)}\right)}$$

$$\partial_\tau W^{DS,redist} \approx \frac{(1-\tau)\nu_e^2\left(1+\frac{\tau}{(1+\eta)(1-\tau)}\right)}{-\frac{\tau}{(1+\eta)(1-\tau)} + (1-\tau)\nu_e^2\left(1+\frac{\tau}{(1+\eta)(1-\tau)}\right)}$$

The corresponding Benabou (or our) shares for this special case are

$$\partial_\tau W^{BEGS,agg.eff} = \frac{-\frac{\tau}{(1+\eta)(1-\tau)}}{-\frac{\tau}{(1+\eta)(1-\tau)} + (1-\tau)\,\nu_e^2}$$

$$\partial_\tau W^{BEGS,redist} = \frac{(1-\tau)\,\nu_e^2}{-\frac{\tau}{(1+\eta)(1-\tau)} + (1-\tau)\,\nu_e^2}$$

So we see that $\partial_\tau W^{DS,agg.eff} \neq \partial_\tau W^{BEGS,agg.eff}$ and $\partial_\tau W^{DS,redist} \neq \partial_\tau W^{BEGS,redist}$. The Davila and Schaab decomposition only agrees with Benabou and us when $\tau = 0$ otherwise it overstate redistribution when $\tau > 0$.

B More Details for Quantitative Application

B.1 Expressions for implementing decomposition in Section

To implement the welfare decomposition outlined in Section 5.1, we need expressions for the quasi-weights $\{\phi_{x,t}\}$ as well as the three components $\{\Gamma_{x,t}, \Delta_{x,t}, \Lambda_{x,t}\}$ for $x \in \{c, n\}$. We list them below:

$$\phi_{c,t}\left(a_0,\epsilon_0\right)=\beta^t\alpha\left(a_0,\epsilon_0\right)\left[c_t^Z\left(a_0,\epsilon_0\right)\right]^{1-\sigma}\qquad \phi_{n,t}\left(a_0,\epsilon_0\right)=-\beta^t\alpha\left(a_0,\epsilon_0\right)\chi\left[n_t^Z\left(a_0,\epsilon_0\right)\right]^{1+\gamma},$$

$$\Gamma_{x,t}=\ln\int\mu_0\left(a_0,\epsilon_0\right)\mathbb{E}\left[x_t^B|a_0,\epsilon_0\right]-\ln\int\mu_0\left(a_0,\epsilon_0\right)\mathbb{E}\left[x_t^A|a_0,\epsilon_0\right],$$

$$\Delta_{x,t}\left(a_0,\epsilon_0\right)=\ln w_{x,t}^B\left(a_0,\epsilon_0\right)-\ln w_{x,t}^A\left(a_0,\epsilon_0\right),$$

$$\Lambda_{x,t}\left(a_0,\epsilon_0\right)=-\frac{1}{2}\left[var\left(\ln x^B|a_0,\epsilon_0\right)-var\left(\ln x^A|a_0,\epsilon_0\right)\right],$$

where objects with superscript Z are harmonic means of their counterparts across $j\in\{A,B\}$.

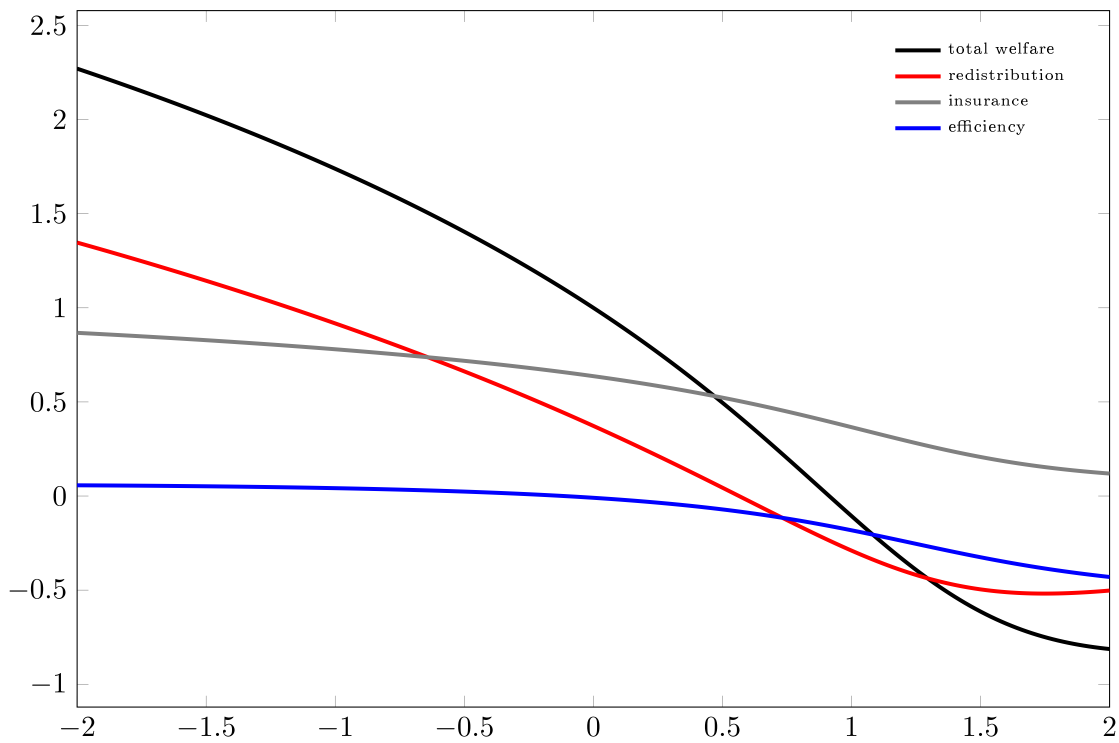
A key object that appears in all the terms above is the future expectations of individual policy variables conditioned on states in date $t=0$, $\mathbb{E}\left[x_t^j|a_0,\epsilon_0\right]$. One could compute $\mathbb{E}\left[x_t^j|a_0,\epsilon_0\right]$ by simulating several paths of shocks and constructing the expectations using Monte Carlo integration. However, this is inefficient and prone to simulation errors. We show how to construct approximations to $\mathbb{E}\left[x_t^j|a_0,\epsilon_0\right]$ directly using approximations to policy functions that are obtained as outcomes of standard methods of solving incomplete market economies.

Let $\left\{\mathrm{m}_0^j,\mathrm{x}_t^j\right\}$ be a vectors that stores the initial distribution (as a histogram) and the policy function x_t on some discrete grid $\mathcal{A}\times\mathcal{E}$; let $\mathrm{H}_t^j\left(a_{t+1},\epsilon_{t+1}|a_t,\epsilon_t\right)$ be a matrix that stores the transition probabilities between date t and $t+1$ for the same discrete grid $\mathcal{A}\times\mathcal{E}$. We solve for stationary equilibrium using the endogenous grid method and solve for the transition paths globally using a Newton algorithm. We use the histogram method to store m_0^j and store $\left\{\mathrm{H}_t^j\right\}$ as sparse matrices. The approximation to $\mathbb{E}\left[x_t^j|a_0,\epsilon_0\right]$ for all pairs $(a_0,\epsilon_0)\in\mathcal{A}\times\mathcal{E}$ can be stored as a column vector $\mathrm{E}_{x,t}^j$ that has length $|\mathcal{A}\times\mathcal{E}|$ and is computed using the matrix product

$$\mathrm{E}_{x,t}^j=\left[\mathrm{H}_1^j\times\mathrm{H}_2^j\times\cdots\mathrm{H}_t^j\right]\times\mathrm{x}_t^j.$$

B.2 Algorithm to implement decomposition 3.3

Suppose we have M points on assets and N points on productivity. We define $s=(a,\epsilon)$ such that $\{(a_0,\epsilon_0),(a_1,\epsilon_0)\ldots(a_M,\epsilon_0)\ldots(a_0,\epsilon_N),\ldots(a_M,\epsilon_N)\}$. This gives an lexicographic order to s with respect to $a,e(\epsilon)$ and we denote it by $\succeq$ to mean $s_{j+1}\succeq s_j$. Define a CDF



Notes: Welfare decomposition across Pareto weights: $\alpha(a_0, \epsilon_0) \propto \exp\{\delta_c c(a_0, \epsilon_0)\}$ with $\delta_c \in [-2, 2]$. The three components of welfare are normalized by $\mathcal{W}^B - \mathcal{W}^A$ at the Utilitarian weights.

$F^j(s)$ using the sum

$$F^j(s) = \sum_{s \succeq \tilde{s}} \mu^j(\tilde{s})$$

For example:

$$F^j(a_0, \epsilon_1) = \mu^j(a_0, \epsilon_0) + \mu^j(a_1, \epsilon_0) + \ldots + \mu^j(a_M, \epsilon_0) + \mu^j(a_0, \epsilon_1)$$

Construct a set $\mathcal{I}$ such that

$$\mathcal{I} \equiv \left\{F^A(s_1), \ldots F^A(s_{M \times N})\right\} \cup \left\{F^B(s_1), \ldots F^B(s_{M \times N})\right\}$$

Now sort $\mathcal{I}$ and store it as $\mathcal{I}^{\text{sorted}} = \text{sort}(\mathcal{I})$. The set $\mathcal{I}^{\text{sorted}}$ constitutes a partition of $[0, 1]$. Every point in $\mathcal{I}^{\text{sorted}}$ gets mapped uniquely a state $s^j(\iota)$ representing the interval in which quantile ι would lie under policy j. Formally $s^j(\iota) = s_k$ such that

$$k = \text{argmin}_{\tilde{k}} \left\{F^j\left(s_{\tilde{k}}\right) \geq \iota\right\}.$$

1. A matrix $\mathrm{T}^j[\iota, s]$ with $|\mathcal{I}^{\text{sorted}}|$ rows and $|\mathcal{A} \times \mathcal{E}|$ columns such that

$$\mathrm{T}^j[\iota, s] = \begin{cases} 1 & \text{if } s^j(\iota) = s \\ 0 & \text{otherwise} \end{cases}$$

   that stores the mapping between $\mathcal{I}^{\text{sorted}}$ and $\mathcal{A} \times \mathcal{E}$. Now we construct the matrix T^j.

2. A vector $\tilde{\mathbf{m}}$ of size $|\mathcal{I}^{\text{sorted}}|$

$$\tilde{\mathbf{m}}[i] = \mathcal{I}^{\text{sorted}}[i] - \mathcal{I}^{\text{sorted}}[i-1]$$

   with the normalization that $\mathcal{I}^{\text{sorted}}[0] = 0$.

The components of equation (13) are given by

$$\begin{aligned}
\text{agg. efficiency} &= \sum_t \sum_\iota \tilde{\mathbf{m}}(\iota)\left[\phi_{c,t}(\iota)\,\Gamma_{c,t} + \phi_{n,t}(\iota)\,\Gamma_{n,t}\right], \\
\text{redistribution} &= \sum_t \sum_\iota \tilde{\mathbf{m}}(\iota)\left[\phi_{c,t}(\iota)\,\Delta_{c,t}(\iota) + \phi_{n,t}(\iota)\,\Delta_{n,t}(\iota)\right], \\
\text{insurance} &= \sum_t \sum_\iota \tilde{\mathbf{m}}(\iota)\left[\phi_{c,t}(\iota)\,\sigma\Lambda_{c,t}(\iota) - \phi_{n,t}(\iota)\,\gamma\Lambda_{n,t}(\iota)\right],
\end{aligned}$$

TABLE 5: WELFARE DECOMPOSITIONS ACROSS RISK AVERSION

	Baseline $\lambda = 10.5$	High Risk aversion $\lambda = 21$
Agg. efficiency	-249.4%	-140.46%
Insurance	149.5%	40.76%
Redistribution	-0.1%	-0.30%

Notes: Component shares of welfare gains for different values of risk aversion paramter(λ). For a given of λ, we divide the welfare gains from the baseline reform attributed to each of the component by absolute value of the sum of all three components.

where

$$w_{x,t}^{j}(\iota) = \frac{\mathbb{E}\left[x_t^j|\iota\right]}{\int \tilde{\mathbf{m}}(\iota)\,\mathbb{E}\left[x_t^j|\iota\right]} \quad x \in \{c, n\},$$

$$\mathbb{E}\left[x_t^j|\iota\right] = \mathrm{T}^j \times \mathrm{H}_{1\to t}^j \times \mathrm{x}_t^j.$$

The rest of the terms are defined as before:

$$w_{x,t}^{Z}(i,\iota) \equiv \sqrt{w_{x,t}^{A}(\iota)\,w_{x,t}^{B}(\iota)} \quad x \in \{c, n\}$$

$$c_t^Z(\iota) = C_t^Z w_{c,t}^Z(\iota)$$

$$n_t^Z(\iota) = N_t^Z w_{n,t}^Z(\iota)$$

$$\phi_{c,t}(\iota) = \beta^t \alpha(\iota)\left[c_t^Z(\iota)\right]^{1-\sigma}$$

$$\phi_{n,t}(\iota) = -\beta^t \alpha(\iota)\,\chi\left[n_t^Z(\iota)\right]^{1+\gamma}$$

$$\Gamma_{c,t} = \ln C_t^B - \ln C_t^A \quad \Gamma_{n,t} = \ln N_t^B - \ln N_t^A$$

$$C_t^Z = \sqrt{C_t^A C_t^B},\ N_t^Z = \sqrt{N_t^A N_t^B}$$

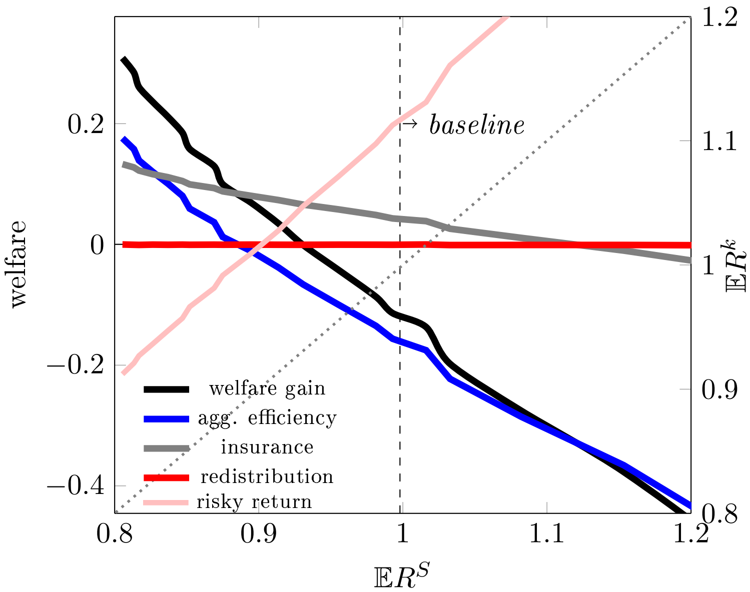
$$\Delta_{x,t}(\iota) = \ln w_{x,t}^B(\iota) - \ln w_{x,t}^A(\iota) \quad x \in \{c, n\}$$

$$\Lambda_{x,t}(\iota) = -\frac{1}{2}\left[var\left(\ln x^B|\iota\right) - var\left(\ln x^A|\iota\right)\right] \quad x \in \{c, n\}$$

B.3 Results for Section 5.2 with alternative calibrations

In the baseline, we set $\lambda = 10.5$ to target a coefficient of relative risk aversion equal to two. In this section, we report results from 5.2 for a higher value of $\lambda = 21$ which correspond to coefficients of relative risk aversions equal to 4. Figure 7 and Table 5 report the results.

Figure 7: WELFARE DECOMPOSITIONS ACROSS DISCOUNT RATES FOR HIGH RISK AVERSION



Notes: Ergodic average of debt returns and and components of welfare across values the discount factor $\beta \in [0.8, 1.2]$ with $\lambda = 21$.